Justification Logic **March 19, 2007**
Instructor: S. Artemov **Take-Home-Exam 1, due March 27**

This is an open book exam. Show all your work. Don't try bonus problems before you are done with the core problems: your grade will be determined by the core problems only!

Problem 1. Which of the de Morgan laws is provable in intuitionistic logic?
a) $\neg(A \vee B) \rightarrow (\neg A \wedge \neg B)$;
b) $(\neg A \wedge \neg B) \rightarrow \neg(A \vee B)$;
c) $\neg(A \wedge B) \rightarrow (\neg A \vee \neg B)$;
d) $(\neg A \vee \neg B) \rightarrow \neg(A \wedge B)$.
Give a proof in $\mathsf{Int}/\mathsf{IntG}$ or a countermodel.

Problem 2. Derive $(\Box A \rightarrow \Box B) \rightarrow \Box(\Box A \rightarrow \Box B)$ in $\mathsf{S5}$. Is this formula derivable in $\mathsf{S4}$?

Problem 3. Show that Grzegorczyk's formula $\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p$ is not provable in $\mathsf{S5}$.

Problem 4. Derive in $\mathsf{S4}$ and realize in LP the following formulas
a) $\Box A \rightarrow \Box(\Box B \rightarrow \Box A)$;
b) $(\Box A \vee \Box B) \rightarrow \Box(A \vee \Box B)$.

Problem 5. Derive in LP and internalize $x{:}(A \vee B),\ A \rightarrow B \vdash B$, i.e., find a proof polynomial $t(x, y)$ such that $x{:}(A \vee B), y{:}(A \rightarrow B) \vdash t(x, y){:}B$. Feel free to use explicit Necessitation rule: if $\vdash F$ then $\vdash s{:}F$ for some ground proof polynomial s.

Bonus Problem 6. Prove that every intuitionistic tautology contains "$\rightarrow$". Hint: consider a cut-free derivation of $\Rightarrow F$.

Bonus Problem 7. Prove the Stripping Rule in LP:

$$\frac{\Gamma, y{:}\Delta \vdash t(y){:}F}{\Gamma, \Delta \vdash F},$$

where y does not occur in $\Gamma, \Delta \Rightarrow F$. Show that this condition cannot be dropped.

Bonus Problem 8. Prove the Abstraction Rule in LP. Let y does not occur in $s{:}\Gamma, A, B$. Then

$$\frac{s{:}\Gamma, y{:}A \vdash t(y){:}B}{s{:}\Gamma \vdash \lambda y.t(y){:}(A \rightarrow B)}$$

for some proof polynomial called $\lambda y.t(y)$ containing no y. Show that the condition 'y does not occur in $s{:}\Gamma, A, B$' cannot be dropped.

Bonus Problem 9. Prove some sort of relevance property for Int: if $\vdash A \rightarrow B$ and A and B have no propositional variables in common, then $\vdash B$ or $\vdash \neg A$. Hint: prove it for classical logic first, you will learn something valuable.