

Exercise 1. Read §1-5 of the paper S. Artemov. ‘Explicit provability and constructive semantics,’ Bulletin of Symbolic Logic, vol. 7, No.1, pp. 1-36, 2001. The paper is also available on the website <http://web.cs.gc.cuny.edu/~sartemov>.

Exercise 2. For the formula F which is

$$(\neg A \vee B) \rightarrow (A \rightarrow B)$$

(here A and B are propositional letters) find its Gödel translation $tr(F)$ and a realization of $tr(F)$ in LP. This will be the formal BHK semantics of F .

Hint: realize both occurrences of $\Box A$ as $x:A$ and both occurrences of $\Box B$ as $y:B$ for simplicity.

Exercise 3. Find an LP-derivation $x:A, B \vdash A \wedge B$ and internalize it, i.e., find a proof polynomial $t(x, y)$ such that $x:A, y:B \vdash t(x, y):(A \wedge B)$.

Exercise 4. Let LP_\emptyset be LP without the rule of axiom necessitation. Prove that any specific derivation in LP can be embedded into LP_\emptyset in the following manner: let $LP \vdash F$ and CS be the list of all formulas $c:A$ introduced this derivation by the axiom necessitation rule. Show that

$$LP_\emptyset \vdash \bigwedge CS \rightarrow F.$$

Exercise 5. The system J4 is LP without the reflexivity axiom $t:F \rightarrow F$.

a) Prove the Internalization Theorem for J4: if $x:\Gamma, \Delta \vdash F$, then $x:\Gamma, y:\Delta \vdash t(x, y):F$ for some proof polynomial $t(x, y)$.

b) Prove the Realization Theorem for K4 in J4: for any K4-derivable modal formula F there is a realization of modalities by proof polynomials of K4 such that the resulting formula F^r is derivable in J4. Hint: take a cut-free derivation of $\Rightarrow F$ in the sequent formulation of K4. Reminder: the only modal rule in the cut-free system for K4 is

$$\frac{\Gamma, \Box \Gamma \Rightarrow F}{\Box \Gamma \Rightarrow \Box F}$$

Exercise 6. Show that there is no LP proof polynomial $?(x)$ such that $\neg x:p \rightarrow ?(x):(\neg x:p)$ (here p is a propositional variable).

Exercise 7. Prove that the Internalization Rule in LP is invertable, i.e., the following Lowering Rule is admissible in LP:

$$\frac{\Gamma, y:\Delta \vdash t(y):F}{\Gamma, \Delta \Rightarrow F},$$

where y does not occur in $\Gamma, \Delta \Rightarrow F$.

Exercise 8. Prove that the Rule of Abstraction is admissible in LP. Let y does not occur in $s:\Gamma, A, B$. Then

$$\frac{s:\Gamma, y:A \vdash t(y):B}{s:\Gamma \vdash \lambda y.t(y):(A \rightarrow B)}$$

for some proof polynomial called $\lambda y.t(y)$.