Justification Logic **February 28, 2007**
Instructor: S. Artemov **Homework 5, due March 13.**

Exercise 1. Read §1-5 of the paper S. Artemov. 'Explicit provability and constructive semantics,' Bulletin of Symbolic Logic, vol. 7, No.1, pp. 1-36, 2001. The paper is also available on the website http://web.cs.gc.cuny.edu/∼sartemov.

Exercise 2. For the following modal formulas find their proofs in S4 and then build realizations of these proofs in LP:
a) $\Box A \to \Box(B \to A)$;
b) $\Box A \to \Box(B \vee \Box A)$;
c) $\Box A \vee \Box B \to \Box(\Box A \vee B)$.
Hint. Derive in S4 first, then try to mimic the S4-derivation in LP. Use '+' to reconcile different proof terms by the same formulas, of needed. One more example: find an explicit version of $\Box A \to \Box(B \to \Box A)$. First, we derive this formula in S4:
1. $\Box A \to (B \to \Box A)$, a propositional axiom;
2. $\Box(\Box A \to (B \to \Box A))$, from 1, by Necessication;
3. $\Box(\Box A \to (B \to \Box A)) \to (\Box\Box A \to \Box(B \to \Box A))$, modal distributivity axiom;
4. $\Box\Box A \to \Box(B \to \Box A)$, by Modus Ponens, from 2,3;
5. $\Box A \to \Box\Box A$, positive introspection axiom;
6. $\Box A \to \Box(B \to \Box A)$, from 4,5, by propositional logic.
Then we emulate this derivation in LP using common sense and some ingenuity.
1. $x{:}A \to (B \to x{:}A)$, a propositional axiom;
2. $c{:}(x{:}A \to (B \to x{:}A))$, by Constant Specification rule R1;
3. $c{:}(x{:}A \to (B \to x{:}A)) \to (!x{:}x{:}A \to (c!x){:}(B \to x{:}A))$, Application Axiom;
4. $!x{:}x{:}A \to (c!x){:}(B \to x{:}A)$, by Modus Ponens, from 2,3;
5. $x{:}A \to !x{:}x{:}A$, proof checking axiom;
6. $x{:}A \to (c!x){:}(B \to x{:}A)$, from 4,5, by propositional logic.

Exercise 3. Prove the deduction theorem for LP: if $\Gamma, A \vdash B$ then $\Gamma \vdash A \to B$.

Exercise 4. Prove Godel's theorem of 1938: if $\mathsf{LP} \vdash F$ then $\mathsf{LP} \vdash p{:}F$ for some proof polynomial p.

Exercise 5. Consider the forgetful projection F^o of a LP-formula F obtained from F by replacing $t{:}G$ to $\Box G$ for any subformula $!t{:}G$. Show that LP^o is a subset of theorems of S4.

Exercise 6. Do you think LP^o actually equals S4? Any proof ideas?

Exercise 7. Let F be a modal tautology $\Box p \to \Box p$.
a) find its realization in LP (i.e. a formula G derivable in LP such that G^{O} is F) OTHER then $t{:}p \to t{:}p$.
b) find a realization of F in LP different from (a).