Justification Logic **February 28, 2007**
Instructor: S. Artemov **Homework 4, due March 6.**

Exercise 1. Prove Gödel's Theorem of 1933:

$$\mathsf{Int} \vdash F \quad \text{yields} \quad \mathsf{S4} \vdash F^{\Box},$$

where $F^{\Box}$ is Gödel's translation consisting in prefixing each occurrence of a subformula in F by $\Box$. Hint: induction on a derivation of F in Int. Base case corresponds to the axioms of Int, prove that their translations are all derivable in $\mathsf{S4}$. The induction step is Modus Ponens in Int.

Exercise 2. Prove McKinsey-Tarski theorem of 1948: for any propositional formula F

$$\mathsf{S4} \vdash F^{\Box} \text{ yields } \mathsf{Int} \vdash F.$$

Hint: suppose $\mathsf{Int} \not\vdash F$. Then there is an intuitionistic Kripke model $\mathcal{K} = (W, \preceq, \Vdash)$ such that F does not hold somewhere in $\mathcal{K}$. Note that $\mathcal{K}$ counts as an $\mathsf{S4}$-model as well, since it is reflexive and transitive. It now suffices to show by induction on a formula that for each subformula G of F at each node x

$$x \Vdash G \quad \Leftrightarrow \quad x \Vdash G^{\Box}\ .$$

Exercise 3. Show that $\mathsf{S4}$ is not the smallest modal logic satisfying Gödel-McKinsey-Tarski theorem. Let $\mathsf{S4}'$ be the logic obtained from $\mathsf{S4}$ by replacing its reflexivity axiom $\Box F \rightarrow F$ by its special case $\Box\Box F \rightarrow \Box F$. Apparently, $\mathsf{S4}' \subseteq \mathsf{S4}$. Show that

$$\mathsf{Int} \vdash F \quad \Leftrightarrow \quad \mathsf{S4}' \vdash F^{\Box}.$$

and that $\mathsf{S4}' \neq \mathsf{S4}$. For the latter find a simple model where all of $\mathsf{S4}'$ hold, but $\Box p \rightarrow p$ does not.

Exercise 4. Show that Gödel's translation maps $\mathsf{S5}$ to classical logic, i.e.,

$$\mathsf{Cl} \vdash F \quad \Leftrightarrow \quad \mathsf{S5} \vdash F^{\Box}.$$

Exercise 5. Establish the Deduction Theorem for K, $\mathsf{K4}$, $\mathsf{S4}$, and $\mathsf{S5}$: if $\Gamma, A \vdash B$ then $\Gamma \vdash A \rightarrow B$. Pick one as an example. Note that in the necessitation rule $\vdash F / \vdash \Box F$ the set of hypotheses is empty.

Exercise 6. Prove the internalization rule in K, $\mathsf{K4}$, $\mathsf{S4}$, and $\mathsf{S5}$:

$$\frac{A_1, A_2, \ldots, A_n \vdash B}{\Box A_1, \Box A_2, \ldots, \Box A_n \vdash \Box B}\ .$$

Exercise 7. Prove Kripke model completeness, finite model property, decidability, equivalence of Hilbert and Gentzen formulations for K. Use the Gentzen formulation with the only non-logical rule

$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ .$$

Use the construction studied in class for $\mathsf{S4}$.

Exercise 8. Consider the system IntG^{+} in the sequent format obtained from IntG by adding new axioms $\Rightarrow \neg\neg A \rightarrow A$. Show that

a) IntG^{+} proves all classical tautologies and only them;

b) Cut cannot be eliminated in IntG^{+}. (Show that the cut-free fragment of IntG^{+} enjoys the Disjunctive Property).