Justification Logic **February 21, 2007**
Instructor: S. Artemov **Homework 3, due Feb. 27**

In what follows by "derive a formula F in S4" we mean either "derive a formula F in the Hilbert-style axiom system for S4" or "derive in S4G a sequent $\Rightarrow F$."

Exercise 1. Prove in S4:
a) $\Box A \rightarrow \Box(B \rightarrow A)$;
b) $\Box A \rightarrow \Box(B \rightarrow \Box A)$;
c) $(\Box A \vee \Box\neg B) \rightarrow \Box(B \rightarrow \Box A)$;
d) $\Box\neg B \rightarrow \Box(\Box B \rightarrow A)$.

Exercise 2. Show that the following formulas are not derivable in S4. To achieve that find an S4-countermodel (i.e. reflexive and transitive countermodel Kripke) for this formula.
a) $\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$;
b) $\Box(\Box A \rightarrow A) \rightarrow \Box A$;
c) $(\Box A \vee \neg\Box B) \rightarrow \Box(\Box B \rightarrow \Box A)$;
d) $A \rightarrow \Box\Diamond A$.

Exercise 3. Which of the following is provable in S4? Give a proof, if any. Provide a countermodel otherwise.
a) $\Box(A \rightarrow \Box B) \rightarrow (A \rightarrow \Box\Box B)$;
b) $\Box(\Box A \rightarrow \Box B) \vee \Box(\Box B \rightarrow \Box A)$.

Exercise 4. Which of the following is provable in S4? in S5?
a) $\Box(p \vee \neg p)$;
b) $\Box p \vee \Box\neg p$;
c) $\Box p \vee \neg\Box p$;
d) $\Box p \vee \Box\neg\Box p$;
e) $(\Box p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)$;
f) $\Box(p \rightarrow q) \rightarrow \Box(\Box p \rightarrow \Box q)$;
g) $\Box(p \rightarrow q) \rightarrow (p \rightarrow \Box q)$;
h) $(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$.

Exercise 5. Which of the following is provable in S5? Give a proof, if any. Provide an-S5 countermodel (i.e. symmetric, reflexive and transitive countermodel Kripke) if a formula in not provable in S5.
a) $\Box(A \rightarrow B) \leftrightarrow (\Box A \rightarrow \Box B)$;
b) $\Box A \leftrightarrow \Box\Box A$;
c) $A \rightarrow \Box A$;
d) $(\Box A \rightarrow \Box B) \vee (\Box B \rightarrow \Box A)$;
e) $\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$;
f) $(\Box A \rightarrow B) \vee (\Box B \rightarrow A)$;
g) $\Box(A \rightarrow B) \vee \Box(B \rightarrow A)$;
h) $\Box(\Box A \rightarrow A) \rightarrow \Box A$;
i) $A \rightarrow \Box\Diamond A$.

Exercise 6. Prove that all the inclusions are strict: $\mathsf{K} \subset \mathsf{K4} \subset \mathsf{S4} \subset \mathsf{S5}$. In practical terms it suffices to show that
$\mathsf{K} \nvdash \Box p \rightarrow \Box\Box p$;
$\mathsf{K4} \nvdash \Box p \rightarrow p$;
$\mathsf{S4} \nvdash \neg\Box p \rightarrow \Box(\neg\Box p)$.

Exercise 7.
a) Establish the Disjunctive Property for S4: $\vdash \Box A \vee \Box B$ yields $\vdash \Box A$ or $\vdash \Box B$.
b) Does the Disjunctive Property hold for S5?