**Justification Logic** **February 15, 2007**
**Instructor: S. Artemov** **Homework 2, due Feb. 20**

**Exercise 1.** In what follows by "derive a formula $F$ in IntG" we mean "derive in IntG a sequent $\Rightarrow F$." Derive in IntG the formulas

a) $(A \wedge B) \rightarrow (B \wedge A)$
b) $(A \vee B) \rightarrow (B \vee A)$
c) $((A \wedge B) \rightarrow C) \leftrightarrow (A \rightarrow (B \rightarrow C))$
d) $\neg(A \wedge \neg A)$
e) $(A \vee B) \rightarrow (\neg A \rightarrow B)$. Is the converse $(\neg A \rightarrow B) \rightarrow (A \vee B)$ derivable?
f ) $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$. Is the converse $(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$ derivable?

**Exercise 2.** Is the Pierce Law $((A \rightarrow B) \rightarrow A) \rightarrow A$ (a classical tautology which does not look like the law of excluded middle at all) derivable in intuitionistic logic? Try to find its derivation in IntG. Describe what happens and make a conclusion.

**Exercise 3.** Prove the atomic axioms observation: using axioms $A \Rightarrow A$ with atomic $A$'s only (i.e. when $A$ is just a propositional letter or $\perp$) does not shrink the set of derivable sequents in IntG. It suffices to derive all the old axioms $F \Rightarrow F$ from the atomic axioms only. Such a proof should go by induction on the complexity of $F$. The base case when $F$ is atomic is covered by the atomic axioms. Consider cases corresponding to all three connectives $\vee, \wedge, \rightarrow$ and prove that if both $X \Rightarrow X$ and $Y \Rightarrow Y$ are derivable, then $(X \alpha Y) \Rightarrow (X \alpha Y)$ is also derivable where $\alpha \in \{\vee, \wedge, \rightarrow\}$.

**Exercise 4.** Derive $\neg\neg(A \vee \neg A)$ in IntG.

**Exercise 5.** Show that $\perp$ cannot be expressed in Int via other connectives $\vee, \wedge, \rightarrow$. Hint: Suppose there is a formula $F(p)$ without $\perp$ such that $\mathsf{Int} \vdash F(p) \leftrightarrow (p \rightarrow \perp)$. Find a contradiction.

**Exercise 6.** Check whether the following is an intuitionsitic tautology:

a) $(p \wedge \neg q) \rightarrow \neg(p \rightarrow q)$
b) $\neg(p \rightarrow q) \rightarrow (p \wedge \neg q)$
c) $((p \vee q) \rightarrow r) \rightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$

**Exercise 7.** Let W be a new axiom schema $\neg A \vee \neg\neg A$. Consider logic IntW obtained by adding W to the list of axioms of Int. Since W is a special case of the law of excluded middle $F \vee \neg F$, the logic IntW is located somewhere in between Int and Cl. Show that $\mathsf{Int} \neq \mathsf{IntW} \neq \mathsf{Cl}$. It clearly suffices to show that $\mathsf{Int} \nvdash$ W and $\mathsf{IntW} \nvdash p \vee \neg p$. Hint: the law of excluded middle $p \vee \neg p$ is true in each one-node model, but has a two-node countermodel. Prove that W is valid in all two-node models, regardless of $A$. This gives you a window of opportunity to distinguish between IntW and Cl.

**Exercise 8.** Let LC be a new axiom schema $(A \rightarrow B) \vee (B \rightarrow A)$. Consider logic LC (logic of linear chains) obtained by adding LC to the list of axioms of Int. Since LC is a classical tautology, the logic LC is located somewhere in between Int and Cl. Show that $\mathsf{Int} \neq \mathsf{LC} \neq \mathsf{Cl}$. It clearly suffices to show that $\mathsf{Int} \nvdash$ LC and LC does not prove some classical tautology. Hint: axiom LC is valid in all linear models.

**Exercise 9.** Show that $\wedge$ cannot be expressed in Int via other connectives $\vee, \rightarrow, \perp$. Hint: consider the Kripke model with $W = (\{0,1,2\}, 0 \prec 2, 1 \prec 2)$ (yes, it is shaped as $\wedge$ itself ), with $0 \Vdash p$, $1 \Vdash q$ and $2 \Vdash p, q$. By induction on a $\wedge$-free formula $F(p,q)$ show that a situation when $0 \nVdash F$, $1 \nVdash F$, and $2 \Vdash F$ is impossible (plenty of cases to be considered, be patient). Note that $p \wedge q$ does just that, i.e. $0 \nVdash p \wedge q$, $1 \nVdash p \wedge q$, and $2 \Vdash p \wedge q$.