Justification Logic **February 7, 2007**
Instructor: S. Artemov **Homework 1, due Feb. 13**

Exercise 1. Suppose a derivation $\Gamma, A \vdash B$ has n lines, i.e. n steps each of which is either invoking an axiom or a hypothesis from $\Gamma \cup \{A\}$, or using the rule *Modus Ponens* once. Give a reasonable upper bound of the number of steps in the derivation $\Gamma \vdash A \rightarrow B$ obtained by applying the proof of the Deduction Theorem above.

Exercise 2. The same question when n is the total number of symbols in the proof $\Gamma, A \vdash B$ and we are interested in an upper bound on the number of symbols in the proof $\Gamma \vdash A \rightarrow B$.

Exercise 3. Prove that $p \vee \neg p$ is not valid in the topological semantics for intuitionistic logic.

Exercise 4. Prove that $p \rightarrow \neg\neg p$ is valid in the topological semantics for intuitionistic logic.

Exercise 5. Prove that $p \vee \neg p$ is not valid in Kripke semantics for intuitionistic logic.

Exercise 6. Show that **Int** $\not\vdash \neg\neg p \rightarrow p$ by finding a countermodel Kripke for this formula (i.e. find a Kripke model $\mathcal{K}$ such that this formula is not forced at some node of $\mathcal{K}$).

Exercise 7. Establish the disjunctive property of **Int**: $\vdash A \vee B$ yields ($\vdash A$ or $\vdash B$). Note that such a property fails for the classical logic where $p \vee \neg p$ is provable, but neither of p nor $\neg p$ is.

Exercise 8. Show that **Int** is not a three-valued logic. In particular, show that the formula

$$(p \leftrightarrow q) \vee (p \leftrightarrow r) \vee (p \leftrightarrow s) \vee (q \leftrightarrow r) \vee (q \leftrightarrow s) \vee (r \leftrightarrow s)$$

is not derivable in **Int**.
Hint: note, that the classical logic **Cl** is two-valued since $(p \leftrightarrow q) \vee (q \leftrightarrow r) \vee (p \leftrightarrow r)$ is a tautology, and thus a theorem of **Cl**. The natural meaning of this formula is that for any three propositions p, q, r at least two of them are equivalent (have the same truth value). In other words, there are no three different truth values. A natural formal representation of the three-valued property of **Int** would be the provability of the dislayed formula above. Show now that the latter formula is not provable in **Int**.

Exercise 9. Prove Glivenko's Theorem (embedding the classical logic into **Int**):

$$\mathbf{Cl} \vdash A \quad \text{iff } \mathbf{Int} \vdash \neg\neg A.$$

The moral here is that intuitionistic logic emulates the classical one (but not the other way around. It takes more than **Cl** to capture **Int**).