Computational Logic, Fall 2002	**Handout 6**
Practice kit for Test 1, Solutions	**October 6, 2002**

Exercise 5.1. Prove the substitution rule for **Int** (which holds for **Cl** as well): If **Int** proves $A \leftrightarrow B$ then for any formula $F(p)$ **Int** proves $F(A) \leftrightarrow F(B)$. Hint: induction on $F(p)$. Base case 1: F is p. Base case 2: F is atom other than p, etc.

In what follows $X \equiv Y$ means that X and Y syntactically coincide.

Base.

a) $F(p) \equiv p$. Then $F(A) \equiv A,\ F(B) \equiv B$ and the induction claim is given by the assumption that **Int** proves $A \leftrightarrow B$.

b) $F(p) \equiv q \not\equiv p$. Then $F(A) \equiv F(B) \equiv q$, trivial.

c) $F(p) \equiv q \not\equiv \bot$ – similar to b).

Induction step. Lemma: $\vdash X \leftrightarrow X'$ and $\vdash Y \leftrightarrow Y'$ yields $\vdash X \wedge Y \leftrightarrow X' \wedge Y'$, $\vdash X \vee Y \leftrightarrow X' \vee Y'$, $\vdash X \to Y \leftrightarrow X' \to Y'$ (an easy exercise).

Case $F(p) \equiv H(p) \wedge G(p)$. By the Induction Hypothesis (I.H.), $\vdash H(A) \leftrightarrow H(B)$ and $\vdash G(A) \leftrightarrow G(B)$. By Lemma above, $\vdash H(A) \wedge G(A) \leftrightarrow H(B) \wedge G(B)$. Other cases: $F(p) \equiv H(p) \vee G(p)$ and $F(p) \equiv H(p) \to G(p)$ are similarly based on the Lemma.

Exercise 5.2. Prove in intutionistic logic using whatever proof tool you wish (**Int**, **IntG**, etc.). Remember that $X \leftrightarrow Y$ stands for $(X \to Y) \wedge (Y \to X)$.

a) $\neg A \leftrightarrow \neg\neg\neg A$

b) $\neg\neg(A \to B) \leftrightarrow (\neg\neg A \to \neg\neg B)$

Solution. a) Was done in early homeworks.

b)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{B \Rightarrow B}{B, A \Rightarrow B}}{A \to B, A \Rightarrow B}}{A \to B, A, \neg B \Rightarrow}}{\neg\neg(A \to B), \neg\neg A, \neg B \Rightarrow}}{\neg\neg(A \to B), \neg\neg A \Rightarrow \neg\neg B}}{\neg\neg(A \to B) \Rightarrow \neg\neg A \to \neg\neg B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\neg A, A \Rightarrow}}{\neg A, A \Rightarrow B}}{\neg A \Rightarrow A \to B}}{\neg(A \to B), \neg A \Rightarrow}}{\neg(A \to B) \Rightarrow \neg\neg A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B}{B, A \Rightarrow B}}{B \to A \to B}}{B, \neg(A \to B) \Rightarrow}}{\neg(A \to B) \Rightarrow \neg B}}{\neg\neg B, \neg(A \to B) \Rightarrow}}{\neg\neg A \to \neg\neg B, \neg(A \to B) \Rightarrow}}{\neg\neg A \to \neg\neg B \Rightarrow \neg\neg(A \to B)}$$

Exercise 5.3. If A does not contain $\vee$ and all propositional letters in A are negated, then **Int** proves $A \leftrightarrow \neg\neg A$. Hint: induction on A, starting with the case A is atomic ...

Base: case $A \equiv p$, given in assumptions that **Int** proves $A \leftrightarrow \neg\neg A$.

Case $A \equiv \bot$. It suffices to prove in **Int** that $\neg\neg\bot \to \bot$:

$$\dfrac{\dfrac{\dfrac{\bot \Rightarrow}{\Rightarrow \neg\bot}}{\neg\neg\bot \Rightarrow}}{\neg\neg\bot \Rightarrow \bot}$$

Induction step. Case $A \equiv (B \wedge C)$: it suffices to prove $\neg\neg(\neg\neg X \wedge \neg\neg Y) \to \neg\neg X \wedge \neg\neg Y$, and even $\neg\neg(\neg\neg X \wedge \neg\neg Y) \to \neg\neg X$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg X \Rightarrow \neg X}{\neg\neg X, \neg X \Rightarrow}}{\neg\neg X \wedge \neg\neg Y, \neg X \Rightarrow}}{\neg\neg(\neg\neg X \wedge \neg\neg Y), \neg X \Rightarrow}}{\neg\neg(\neg\neg X \wedge \neg\neg Y) \Rightarrow \neg\neg X}$$

Case $A \equiv (B \to C)$ is covered by 5.2b.

Exercise 5.4. This suggests another embedding of the classical logic to the intuitionistic one. Define Gödel's translation φ^o of a given propositional formula φ: get rid of $\vee$ by the classical de Morgan Law $A \vee B := \neg(\neg A \wedge \neg B)$ and prefix all propositional letters by $\neg\neg$. The resulting φ^o is clearly equivalent to φ classically. Show that **Cl** proves φ iff **Int** proves φ^o.

Proof. As noticed, **Cl** proves $\varphi \leftrightarrow \varphi^o$. Since **Cl** proves φ, **Cl** proves φ^o. By Glivenko's Theorem, **Int** proves $\neg\neg\varphi^o$. By 5.3, **Int** proves φ^o. The direction **Int** proves φ^o yields **Cl** proves φ is trivial.

Exercise 5.5. Prove in **Int**:
a) $\neg(A \vee B) \leftrightarrow \neg(\neg A \to B)$
b) $\neg\neg(A \vee B) \leftrightarrow \neg\neg(\neg A \to B)$

Since (b) is an obvious corollary of (a), we will prove (a).

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A \vee B}}{\neg(A \vee B) \Rightarrow \neg A} \qquad \dfrac{\dfrac{B \Rightarrow B}{B \Rightarrow A \vee B}}{\neg(A \vee B), B \Rightarrow}}{\neg(A \vee B), \neg A \to B \Rightarrow}}{\neg(A \vee B) \Rightarrow \neg(\neg A \to B)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, \neg A \Rightarrow}}{A, \neg A \Rightarrow B} \qquad \dfrac{B \Rightarrow B}{B, \neg A \Rightarrow B}}{A \vee B, \neg A \Rightarrow B}}{\neg(\neg A \to B), A \vee B \Rightarrow}}{\neg(\neg A \to B) \Rightarrow \neg(A \vee B)}$$

Exercise 5.6. Yet another embedding of **Cl** into **Int**. Define Kolmogorov's translation $\varphi^{\neg\neg}$ of φ consisting in prefixing all subformulas of φ by $\neg\neg$. Show that **Cl** proves φ iff **Int** proves $\varphi^{\neg\neg}$.

Proof. A general reasoning schema first. Let $t(F)$ denote the result of replacing $A \vee B$ by $\neg A \to B$ everywhere in F. Then
1. $\mathbf{Cl} \vdash \varphi$, thus $\mathbf{Cl} \vdash t(\varphi)$, and thus $\mathbf{Cl} \vdash [t(\varphi)]^{\neg\neg}$ by trivial reasons. Note, that $t(\varphi)$ is $\vee$-free.
2. $\mathbf{Int} \vdash \neg\neg[t(\varphi)]^{\neg\neg}$, by Glivenko's Theorem.
3. By 5.3, $\mathbf{Int} \vdash [t(\varphi)]^{\neg\neg}$.
4. It remains to show that $\mathbf{Int} \vdash \varphi^{\neg\neg} \leftrightarrow [t(\varphi)]^{\neg\neg}$ and we are done. The latter goes by a straightforward induction on (the length of) φ with the use of 5.5 at the induction step.

Base: φ is atomic is trivial, since in this case $\varphi^{\neg\neg}$ and $[t(\varphi)]^{\neg\neg}$ coincide.

Induction step. The only nontrivial case is the case of disjunction: φ is $A \vee B$. In what follows $X \overset{\mathbf{Int}}{\longleftrightarrow} Y$ means $\mathbf{Int} \vdash X \leftrightarrow Y$. Note, that by I.H., $A^{\neg\neg} \overset{\mathbf{Int}}{\longleftrightarrow} [t(A)]^{\neg\neg}$ and $B^{\neg\neg} \overset{\mathbf{Int}}{\longleftrightarrow} [t(B)]^{\neg\neg}$.

$[t(\varphi)]^{\neg\neg} \equiv \neg\neg(\neg\neg\neg[t(A)]^{\neg\neg} \to [t(B)]^{\neg\neg}) \overset{\mathbf{Int}}{\longleftrightarrow} \neg\neg(\neg[t(A)]^{\neg\neg} \to [t(B)]^{\neg\neg}) \overset{\mathbf{Int}}{\longleftrightarrow}$ (by 5.5)$\neg\neg([t(A)]^{\neg\neg} \vee [t(B)]^{\neg\neg}) \overset{\mathbf{Int}}{\longleftrightarrow}$ (by I.H.)$\neg\neg(A^{\neg\neg} \vee B^{\neg\neg}) \equiv \varphi^{\neg\neg}$

Exercise 5.7. Show that the Disjunctive Property with hypotheses does not hold in **Int**: $\Gamma \vdash A \vee B$ not necessarily yields $\Gamma \vdash A$ or $\Gamma \vdash B$. Hint: find an easy counterexample.

Solution. In **Int** $A \vee B \vdash A \vee B$, but neither $A \vee B \vdash A$, nor $A \vee B \vdash B$

Exercise 5.8. There is no formula F without $\rightarrow$ such that **Int** proves F (note that negation $\neg A$ is a shorthand for $A \rightarrow \bot$ and thus does contain an implication $\rightarrow$). Hint: analyze a possible cut-free derivation of $\Rightarrow F$ (there are other natural solutions as well).

Solution. Consider a cut free proof of $\Rightarrow F$ (which exists by the cut elimination theorem). Since the antecedent (the part to the left of $\Rightarrow$) is empty, there should be rules in the derivation that move formulas from the antecedent to the succedent. The only such rule is ($\Rightarrow$,$\rightarrow$) that introduces $\rightarrow$. Once introduced, this "$\rightarrow$" should appear in the final sequent, by the subformula property of cut-free proofs.

Exercise 5.9. Show that there are infinitely many logics stronger than **Int** but still weaker than **Cl**. Hint: consider formulas

$$I_n = \bigvee_{0 \leq i < j \leq n} (p_i \leftrightarrow p_j)$$

and logics $\mathbf{Int}_n = \mathbf{Int} + I_n$ (i.e. axioms of **Int** with additional axiom I_n and rule MP), $n \geq 2$. Show that for all such n's logics $\mathbf{Int}_n$ are pairwise different and all fall in between **Cl** and **Int**.

Solution. First, we notice that each of $\mathbf{Int}_n$ ($n > 2$) is indeed in between **Int** and **Cl**. The fact that **Int** is contained in $\mathbf{Int}_n$ is obvious by the construction of $\mathbf{Int}_n$ which is defined as **Int**+something else. On the other hand, obviously $\mathbf{Int}_n$ is a sublogic of **Cl**, since all **Int** and I_n are tautologies and thus belong to **Cl**. What remains to show is that all $\mathbf{Int}_n$ are pairwise distinct.

To show this build an obvious series of Kripke models K_n such that $K_n \models I_m$ for all $m > n$ and $K_n \not\models I_n$. Conclude that $\mathbf{Int}_n$ is different from $\mathbf{Int}_m$ for all $m \neq n$. Indeed, assume that $m > n$. Then $\mathbf{Int}_m$ does not prove I_n (an axiom of $\mathbf{Int}_n$), since all axioms and thus all theorems of $\mathbf{Int}_m$ hold in K_n, but I_n does not.