

Make sure you have mastered all the ideas contained in exercises from HO2, HW2 and HW3. Solutions to HW3 will be posted very soon.

Exercise 5.1. Prove the substitution rule for **Int** (which holds for **Cl** as well): If **Int** proves $A \leftrightarrow B$ then for any formula $F(p)$ **Int** proves $F(A) \leftrightarrow F(B)$. Hint: induction on $F(p)$. Base case 1: F is p . Base case 2: F is atom other than p , etc.

Exercise 5.2. Prove in intuitionistic logic using whatever proof tool you wish (**Int**, **IntG**, etc.). Remember that $X \leftrightarrow Y$ stands for $(X \rightarrow Y) \wedge (Y \rightarrow X)$.

- a) $\neg A \leftrightarrow \neg\neg\neg A$
- b) $\neg\neg(A \rightarrow B) \leftrightarrow (\neg\neg A \rightarrow \neg\neg B)$

Exercise 5.3. If A does not contain \vee and all propositional letters in A are negated, then **Int** proves $A \leftrightarrow \neg\neg A$. Hint: induction on A , starting with the case A is atomic ...

Exercise 5.4. This suggests another embedding of the classical logic to the intuitionistic one. Define Gödel's translation φ^o of a given propositional formula φ : get rid of \vee by the classical de Morgan Law $A \vee B := \neg(\neg A \wedge \neg B)$ and prefix all propositional letters by \neg . The resulting φ^o is clearly equivalent to φ classically. Show that **Cl** proves φ iff **Int** proves φ^o .

Exercise 5.5. Prove in **Int**:

- a) $\neg(A \vee B) \leftrightarrow \neg(\neg A \rightarrow B)$
- b) $\neg\neg(A \vee B) \leftrightarrow \neg\neg(\neg A \rightarrow B)$

Exercise 5.6. Yet another embedding of **Cl** into **Int**. Define Kolmogorov's translation $\varphi^{\neg\neg}$ of φ consisting in prefixing all subformulas of φ by \neg . Show that **Cl** proves φ iff **Int** proves $\varphi^{\neg\neg}$.

Exercise 5.7. Show that the Disjunctive Property with hypotheses does not hold in **Int**: $\Gamma \vdash A \vee B$ not necessarily yields $\Gamma \vdash A$ or $\Gamma \vdash B$. Hint: find an easy counterexample.

Exercise 5.8. There is no formula F without \rightarrow such that **Int** proves F (note that negation $\neg A$ is a shorthand for $A \rightarrow \perp$ and thus does contain an implication \rightarrow). Hint: analyze a possible cut-free derivation of $\Rightarrow F$ (there are other natural solutions as well).

Exercise 5.9. Show that there are infinitely many logics stronger than **Int** but still weaker than **Cl**. Hint: consider formulas

$$I_n = \bigvee_{0 \leq i < j \leq n} (p_i \leftrightarrow p_j)$$

and logics $\mathbf{Int}_n = \mathbf{Int} + I_n$ (i.e. axioms of **Int** with additional axiom I_n and rule MP), $n \geq 2$. Show that for all such n 's logics \mathbf{Int}_n are pairwise different and all fall in between **Cl** and **Int**.