Computational Logic, Fall 2002	**Handout 2**
Exercises for Lecture 2	**September 11, 2002**

Reading: van Dalen *Logic and Structure*, Lecture 2 slides.

Exercise 2.1. Suppose a derivation $\Gamma, A \vdash B$ has n lines, i.e. n steps each of which is either invoking and axiom or a hypothesis from Γ, invoking A, or using the rule Modus Ponens once. Give a reasonable upper bound of the number of steps in the derivation $\Gamma \vdash A \to B$ obtained by applying the proof of the Deduction Theorem (lecture 2).

Solution. Let us inspect each possible step of the first derivation and figure our how many steps of the new derivation it produces according to the proof of DT.

Step 1. Picking F which is an axiom or an element of Γ. The corresponding steps of the the new derivations are

F, as an axiom or a hypothesis from Γ

$F \to (A \to F)$, axiom 1

$A \to F$, by MP

As we can see here one old step produces three new ones.

Step 2. Picking A. In the new derivation this will be replaced by five line mini-derivation of $A \to A$ (lecture 2, slide 8).

Step 3. Applying Modus Ponens to $C \to F$ and C. This step will produce the following three lines of the new derivations.

$(A \to (C \to F)) \to ((A \to C) \to (A \to F))$, axiom 2

$(A \to C) \to (A \to F)$, by MP from the above and $A \to (C \to F)$ derived earlier (by the Induction Hypothesis).

$A \to F$

We can see that each step of the old derivation produced at most five steps of the new one. Answer: the number of lines in the new proof is $\leq 5n$ (a linear expansion).

Exercise 2.2. Prove that $p \vee \neg p$ is not valid in the topological semantics for intuitionistic logic.

Solution. It suffices to consider the topology on the real line $\mathbf{R}$. Pick evaluation $t(p)$ of p as the open set $\mathbf{R} - \{0\}$, i.e. the line without one point 0. According to definition (slide 20, lecture 2), the value of formula $p \vee \neg p$ is an open set $t(p) \cup interior(\overline{t(p)})$. Since $t(p) = \mathbf{R} - \{0\}$, $\overline{t(p)} = \{0\}$ and $interior(\overline{t(p)}) = \emptyset$. Thus $t(p) \cup interior(\overline{t(p)}) = t(p) \cup \emptyset = T(P) = \mathbf{R} - \{0\} \neq \mathbf{R}$.

Exercise 2.3. Prove that $p \to \neg\neg p$ is valid in the topological semantics for intuitionistic logic.

Solution. Suppose A has been evaluated as an open set X in some topological space (think $\mathbf{R}$). Then

$interior(\overline{X}) \subseteq \overline{X}$, since the interior of a set is included in this set

$\overline{\overline{X}} \subseteq \overline{interior(\overline{X})}$, compliments of the both sides are taken

$X \subseteq \overline{interior(\overline{X})}$, since double compliments over X collapse.

$interior(X) \subseteq interior(\overline{interior(\overline{X})})$, taken interiors of both sides

$X \subseteq interior(\overline{interior(\overline{X})})$, since X is open and thus $X = interior(X)$

Now note that $interior(\overline{interior(\overline{X})})$ is the value of $t(\neg\neg A)$. Therefore, $t(A) \subseteq t(\neg\neg A)$, thus

$\overline{t(A)} \cup t(\neg\neg A)$ is the whole space, so is $interior(\overline{t(A)} \cup t(\neg\neg A))$, which is the intuitionistic value of $p \to \neg\neg p$.

Well, it seems one has to be pretty fluent in elementary topology to work with this kind of semantics.

Exercise 2.4 Prove that $p \to \neg\neg p$ is valid in Kripke semantics for intuitionistic logic.

Solution. Suppose on a way to contradiction, that at some node x of some Kripke model K the formula $p \to \neg\neg p$ does not hold (i.e. $x \not\models p \to \neg\neg p$). Then at some other node y (accessible from x)

a) $y \models p$ and

b) $y \not\models \neg\neg p$.

By (a), p holds at each world accessible from y (monotonicity!). By (b), there should be a node z accessible from y such where $\neg p$ holds. In particular, $z \not\models p$ - a contradiction.

Exercise 2.5. Show that **Int** $\not\vdash p \vee \neg p$ by finding a countermodel Kripke for this formula (i.e. find a Kripke model K such that this formula is not forced at some node of K).

Solution. Take $K = (W, \preceq, \models)$ where $W = \{0, 1\}$, $0 \preceq 0, 0 \preceq 1, 1 \preceq 1$, $1 \models p$, $0 \not\models p$. Then $0 \not\models \neg p$, since there is an accessible world (namely 1) where p holds. Since $0 \not\models p$ as well, $0 \not\models p \vee \neg p$.

Exercise 2.6. Establish the disjunctive property of **Int**: $\vdash A \vee B$ yields ($\vdash A$ or $\vdash B$).

Solution. Let us assume that **Int** proves $A \vee B$ but neither $\vdash A$ nor $\vdash B$. By the completeness theorem for **Int** with respect to Kripke semantics, there are Kripke models K_A and K_B where A and B are not true respectively. Construct a new model K by adding a new common node 0 "below" each of the models K_A and K_B. Assume that all propositional atoms are false at 0, which guarantees the monotonicity condition on K. We claim that $0 \not\models A \vee B$ in K. Indeed, if $0 \models A \vee B$, then $0 \models A$ or $0 \models B$. In the first of those cases A should be true throughout the whole model K, in particular, in all K_A, which is impossible by the choice of K_A. Likewise, if $0 \models B$, then B should hold everywhere in K, in particular, in K_B, which is also impossible.

Exercise 2.7. Show that **Int** is not a three valued logic. In particular, show that the formula $(p \leftrightarrow q) \vee (p \leftrightarrow r) \vee (p \leftrightarrow s) \vee (q \leftrightarrow r) \vee (q \leftrightarrow s) \vee (r \leftrightarrow s)$ is not derivable in **Int**.

Solution. Otherwise, by the Disjunctive property at least one of the equivalences, e.g. $p \leftrightarrow q$ would be derivable in **Int**, which is not the case ($p \leftrightarrow q$ is not even a classical tautology).