

Exercise 16.1.

a) Derive in \rightarrow **IntH**: $((B \rightarrow A) \rightarrow A) \rightarrow A \vdash B \rightarrow A$. Feel free to use the Deduction Theorem.

Solution. 1. B , a hypothesis
 2. $B \rightarrow A$, a hypothesis
 3. A , from 1,2, by MP
 4. $(B \rightarrow A) \rightarrow A$, by Deduction Theorem, disregarding 2
 5. $((B \rightarrow A) \rightarrow A) \rightarrow A$, a hypothesis
 6. A , by MP, from 4,5
 7. $B \rightarrow A$, by Deduction Theorem, disregarding 1.

b) Internalize the deduction from (a) as a typed combinatory term. Feel free to use emulated λ -abstraction. There is no need to unfold it back to the standard combinatory format, unless you have to.

Solution. 1. $x:B$, a hypothesis
 2. $y:(B \rightarrow A)$, a hypothesis
 3. $yx:A$, from 1,2, by MP
 4. $\lambda^*y.yx:((B \rightarrow A) \rightarrow A)$, by emulated λ -abstraction
 5. $z:(((B \rightarrow A) \rightarrow A) \rightarrow A)$, a hypothesis
 6. $z(\lambda^*y.yx):A$, by MP, from 4,5
 7. $\lambda^*x.z(\lambda^*y.yx):(B \rightarrow A)$, by Deduction Theorem, disregarding 1.

c) Check whether the combinatory term from (b) is normal.

Solution. The term from 7 is normal as a λ -term. It is easy to see that in such a case the result of decoding emulated abstraction into a pure combinatory language will create no redexes. Hence 7 as a combinatory term is also normal.

Exercise 16.2. Find a combinator **2** such that $\mathbf{2}FA = F(FA)$ for all combinators F and A . If you use λ -abstraction, convert the result into a purely combinatory form. Is **2** a normal form?

Solution. We try to construct **2** as a λ -term first. $\mathbf{2}FA = (\mathbf{2}F)A = F(FA) = [\lambda x.F(Fx)]A = [(\lambda yx.y(yx))F]A$. Thus it suffices to take $\mathbf{2} = \lambda yx.y(yx)$. This term is normal. Converting it back to a purely combinatory format is straightforward, we will skip it here. Do not worry, I will not ask you to do a big mechanical work (like the conversion we have just skipped) on the test.

Exercise 16.3. Write completely a λ -term whose normal form is the Church numeral $\overline{100}$.

Solution. According to Lecture 9, HW 7, and T&S 1.2.20, the “multiplication” is represented by term $t = \lambda xyz.x(yz)$ in the sense that $t \cdot \bar{n} \cdot \bar{m} = \overline{n \times m}$ for all Church numerals \bar{n} and \bar{m} . A possible concise though complete presentation of $\overline{100}$ is $t \cdot \bar{4} \cdot (t \cdot \bar{5} \cdot \bar{5})$, which is

$$\lambda xyz.x(yz) \cdot \lambda uz.u(u(u(uz))) \cdot (\lambda xyz.x(yz) \cdot \lambda uz.u(u(u(uz)))) \cdot \lambda uz.u(u(u(uz))))$$

Exercise 16.4. Prove that all the inclusions are strict: $\mathbf{K} \subset \mathbf{K4} \subset \mathbf{S4} \subset \mathbf{S5}$. In practical terms it suffices to show that

- $\mathbf{K} \not\vdash \Box p \rightarrow \Box \Box p$,
- $\mathbf{K4} \not\vdash \Box p \rightarrow p$,
- $\mathbf{S4} \not\vdash \neg \Box p \rightarrow \Box \neg \Box p$.

Solution. Naturally, we rely on the Soundness Theorem here and build corresponding countermodels.

1. **K** $\not\models \Box p \rightarrow \Box \Box p$. Countermodel (no restrictions on the accessibility relation): $W = \{0, 1, 2\}$, $R = \{(0, 1), (1, 2)\}$, $1 \models p$. Then $2 \not\models p$, $1 \not\models \Box p$, $0 \not\models \Box \Box p$. On the other hand, $1 \models p$, thus $0 \models \Box p$ (1 is the only node accessible from 0), thus $0 \not\models \Box p \rightarrow \Box \Box p$.

2. **K4** $\not\models \Box p \rightarrow p$. The countermodel should be transitive, and here it is: $W = \{0, 1\}$, $R = \{(0, 1)\}$, $1 \models p$. In this model $0 \models \Box p$, and $0 \not\models p$.

3. **S4** $\not\models \neg \Box p \rightarrow \Box \neg \Box p$. The countermodel here should be reflexive and transitive; $W = \{0, 1\}$, $R = \{(0, 0), (0, 1), (1, 1)\}$, $1 \models p$. Here $1 \models \Box p$, $1 \not\models \neg \Box p$, $0 \not\models \Box \neg \Box p$. On the other hand, $0 \not\models p$, $0 \not\models \Box p$ (since 0 is accessible from 0), thus $0 \models \neg \Box p$ and $0 \not\models \neg \Box p \rightarrow \Box \neg \Box p$.

Exercise 16.5. Which of the following is provable in **S4**? in **S5**?

- $\Box(p \vee \neg p)$
- $\Box p \vee \Box \neg p$
- $\Box p \vee \neg \Box p$
- $\Box p \vee \Box \neg \Box p$
- $(\Box A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$
- $\Box(A \rightarrow B) \rightarrow \Box(\Box A \rightarrow \Box B)$
- $\Box(A \rightarrow B) \rightarrow (A \rightarrow \Box B)$
- $(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

Solution.

a) $\Box(p \vee \neg p)$. Both, as a Necessitation of a tautology.

b) $\Box p \vee \Box \neg p$. None, a countermodel: $W = \{0, 1\}$, R is total on W (each node is accessible from each), $1 \models p$. It is an **S5**-model, i.e. reflexive, symmetric, and transitive, and $0 \not\models p$, hence $0 \not\models \Box p$. On the other hand, $0 \not\models \Box \neg p$ either, since $1 \models p$.

c) $\Box p \vee \neg \Box p$. Both, as a tautology.

d) $\Box p \vee \Box \neg \Box p$. This is equivalent to $\neg \Box p \rightarrow \Box \neg \Box p$, which is an axiom of **S5** (hence provable in **S5**) not provable in **S4**, by 16.4.3 above.

e) $(\Box A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$. None. An **S5**-countermodel: $W = \{0, 1\}$, R is total on W , $0 \models A$. Here $0 \models \Box A \rightarrow \Box B$, since both $\Box A$ and $\Box B$ are false at 0. On the other hand, $0 \not\models A \rightarrow B$, hence $0 \not\models \Box(A \rightarrow B)$.

f) $\Box(A \rightarrow B) \rightarrow \Box(\Box A \rightarrow \Box B)$. Both. Here is a derivation:

$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$, distributivity axiom;

$\Box(\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B))$, Necessitation;

$\Box \Box(A \rightarrow B) \rightarrow \Box(\Box A \rightarrow \Box B)$, by distributivity;

$\Box A \rightarrow \Box \Box A$, transitivity axiom;

$\Box(A \rightarrow B) \rightarrow \Box(\Box A \rightarrow \Box B)$, by propositional logic from two previous lines.

g) $\Box(A \rightarrow B) \rightarrow (A \rightarrow \Box B)$. None. An **S5**-countermodel: $W = \{0, 1\}$, R is total on W , $0 \models A$ and $0 \models B$. Clearly, $0 \models A \rightarrow B$ and $1 \models A \rightarrow B$, hence, $0 \models \Box(A \rightarrow B)$. On the other hand, $0 \not\models \Box B$, $0 \not\models A \rightarrow \Box B$, thus $0 \not\models \Box(A \rightarrow B) \rightarrow (A \rightarrow \Box B)$.

h) $(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$. None. An **S5**-countermodel: $W = \{0, 1\}$, R is total on W , $0 \models A$, $0 \models B$ and $1 \models A$. Here $0 \models A \rightarrow B$, but $0 \not\models \Box A \rightarrow \Box B$, since $0 \models \Box A$, but $0 \not\models \Box B$.

Exercise 16.6. Show that **S5**=**S4**+ $\Box(A \rightarrow \neg \Box B) \rightarrow (A \rightarrow \Box \neg \Box B)$. Basically, you have to demonstrate that both

- S5** $\vdash \Box(A \rightarrow \neg \Box B) \rightarrow (A \rightarrow \Box \neg \Box B)$ and
- S4**+ $\Box(A \rightarrow \neg \Box B) \rightarrow (A \rightarrow \Box \neg \Box B) \vdash \neg \Box A \rightarrow \Box \neg \Box A$.

Solution. Derivation (a) in **S5**:

$\Box(A \rightarrow \neg \Box B) \rightarrow (A \rightarrow \neg \Box B)$, reflexivity axiom;

$\neg \Box B \rightarrow \Box \neg \Box B$, negative introspection;

$(A \rightarrow \neg \Box B) \rightarrow (A \rightarrow \Box \neg \Box B)$, from the previous line, by propositional logic;
 $\Box(A \rightarrow \neg \Box B) \rightarrow (A \rightarrow \Box \neg \Box B)$, by syllogism from the first and the last lines.

Derivation (b) in **S4**+ $\Box(A \rightarrow \neg \Box B) \rightarrow (A \rightarrow \Box \neg \Box B)$:
 $\neg \Box A \rightarrow \neg \Box A$, a tautology;
 $\Box(\neg \Box A \rightarrow \neg \Box A)$, by Necessitation;
 $\Box(\neg \Box A \rightarrow \neg \Box A) \rightarrow (\neg \Box A \rightarrow \Box \neg \Box A)$, a “new” axiom above;
 $\neg \Box A \rightarrow \Box \neg \Box A$, by MP.

Exercise 16.7. Prove in **S4**:

- a) $\Box A \rightarrow \Box(B \rightarrow A)$
- b) $\Box A \rightarrow \Box(B \rightarrow \Box A)$
- c) $(\Box A \vee \Box \neg B) \rightarrow \Box(B \rightarrow \Box A)$
- d) $\Box \neg B \rightarrow \Box(\Box B \rightarrow A)$

Solution.

- a) $\Box A \rightarrow \Box(B \rightarrow A)$
 - 1. $A \rightarrow (B \rightarrow A)$, an axiom;
 - 2. $\Box(A \rightarrow (B \rightarrow A))$, by necessitation;
 - 3. $\Box A \rightarrow \Box(B \rightarrow A)$, by distribution
- b) $\Box A \rightarrow \Box(B \rightarrow \Box A)$
 - 1. $\Box A \rightarrow (B \rightarrow \Box A)$, an axiom;
 - 2. $\Box(\Box A \rightarrow (B \rightarrow \Box A))$, by necessitation;
 - 3. $\Box \Box A \rightarrow \Box(B \rightarrow \Box A)$, by distribution;
 - 4. $\Box A \rightarrow \Box \Box A$, transitivity axiom;
 - 5. $\Box A \rightarrow \Box(B \rightarrow \Box A)$, by syllogism, from 3,4
- c) $(\Box A \vee \Box \neg B) \rightarrow \Box(B \rightarrow \Box A)$
 - 1. $\Box A \rightarrow \Box(B \rightarrow \Box A)$, as above;
 - 2. $\neg B \rightarrow (B \rightarrow \Box A)$, in propositional logic;
 - 3. $\Box(\neg B \rightarrow (B \rightarrow \Box A))$, by necessitation;
 - 4. $\Box \neg B \rightarrow \Box(B \rightarrow \Box A)$, by distribution;
 - 5. $(\Box A \vee \Box \neg B) \rightarrow \Box(B \rightarrow \Box A)$, from 1,4 in propositional logic.
- d) $\Box \neg B \rightarrow \Box(\Box B \rightarrow A)$
 - 1. $\Box B \rightarrow B$, reflexivity axiom;
 - 2. $\neg B \rightarrow \neg \Box B$, contrapositive of 1.;
 - 3. $\neg \Box B \rightarrow (\Box B \rightarrow A)$, a propositional tautology;
 - 4. $\neg B \rightarrow (\Box B \rightarrow A)$, from 1,3, by syllogism;
 - 5. $\Box(\neg B \rightarrow (\Box B \rightarrow A))$, by Necessitation, from 4;
 - 6. $\Box \neg B \rightarrow \Box(\Box B \rightarrow A)$, by distributivity of \Box through \rightarrow .

Exercise 16.8. Show that the following formulas are not derivable in **S4**. To achieve that find an **S4**-countermodel (i.e. reflexive and transitive countermodel Kripke) for this formula.

- a) $\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$
- b) $\Box(\Box A \rightarrow A) \rightarrow \Box A$.
- c) $(\Box A \vee \neg \Box B) \rightarrow \Box(\Box B \rightarrow \Box A)$
- d) $A \rightarrow \Box \Diamond A$

Solution.

a) $W = \{0, 1, 2\}$, $R(0, 1), R(0, 2)$ + reflexivity, $1 \models A$, $2 \models B$. Then $1 \not\models \Box A \rightarrow B$, $2 \not\models \Box B \rightarrow A$, and

$0 \not\models \Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$.

b) $W = \{0\}$, $R(0,0)$, $0 \not\models A$.

a) $W = \{0,1\}$, $R = \{(0,0), (0,1), (1,1)\}$, $1 \models B$. Then $0 \not\models B$, thus $0 \models \neg\Box B$ and $0 \models \Box A \vee \neg\Box B$. On the other hand, $1 \models \Box B$ but $1 \not\models \Box A$, thus $1 \not\models \Box B \rightarrow \Box A$, and $0 \not\models \Box(\Box B \rightarrow \Box A)$.

b) $W = \{0,1\}$, $R = \{(0,0), (0,1), (1,1)\}$, $0 \models A$. Then $1 \not\models A$, $1 \not\models \Diamond A$ (since only 1 is accessible from 1), $0 \not\models \Box\Diamond A$.

Exercise 16.9. Which of the following is provable in **S5**? Give a proof, if any. Provide an-**S5** countermodel (i.e. symmetric, reflexive and transitive countermodel Kripke) for a formula.

- a) $\Box(A \rightarrow B) \leftrightarrow (\Box A \rightarrow \Box B)$
- b) $\Box A \leftrightarrow \Box\Box A$
- c) $A \rightarrow \Box A$
- d) $(\Box A \rightarrow \Box B) \vee (\Box B \rightarrow \Box A)$
- e) $\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$
- f) $\Box(\Box A \rightarrow A) \rightarrow \Box A$
- g) $(\Box A \rightarrow B) \vee (\Box B \rightarrow A)$
- h) $\Box(A \rightarrow B) \vee \Box(B \rightarrow A)$
- i) $A \rightarrow \Box\Diamond A$

Solution.

a) Not provable. $W = \{0,1\}$, R is the total relation on W , $0 \models A$. Then $0 \not\models A \rightarrow B$ and in each of $\{0,1\}$ all three formulas $\Box(A \rightarrow B)$, $\Box A$, $\Box B$ are false.

b) Provable, easy.

c) Not provable, the same countermodel as in (a).

d) This is an instance of propositional tautology $(p \rightarrow q) \vee (q \rightarrow p)$, provable in the propositional logic.

e) Provable in **S5**. We first establish that $(\Box A \rightarrow \Box B) \rightarrow \Box(\Box A \rightarrow B)$. Indeed,

1. $\neg\Box A \rightarrow \Box(\neg\Box A)$, axiom **S5**;
2. $\neg\Box A \rightarrow (\Box A \rightarrow B)$, a propositional tautology;
3. $\Box[\neg\Box A \rightarrow (\Box A \rightarrow B)]$, by necessitation;
4. $\Box(\neg\Box A) \rightarrow \Box(\Box A \rightarrow B)$, by distribution;
5. $\neg\Box A \rightarrow \Box(\Box A \rightarrow B)$, by syllogism, from 1,4;
6. $B \rightarrow (\Box A \rightarrow B)$, propositional axiom;
7. $\Box B \rightarrow \Box(\Box A \rightarrow B)$, *Necessitation* + distribution;
8. $(\neg\Box A \vee \Box B) \rightarrow \Box(\Box A \rightarrow B)$, from 5,7;
9. $(\Box A \rightarrow \Box B) \rightarrow \Box(\Box A \rightarrow B)$, by propositional logic.

Likewise, we note that $(\Box B \rightarrow \Box A) \rightarrow \Box(\Box B \rightarrow A)$. Therefore,

$$[(\Box A \rightarrow \Box B) \vee (\Box B \rightarrow \Box A)] \rightarrow [\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)].$$

Now apply d.

f) Not provable. A countermodel as in 16.8b).

g) Yes, it is even provable in **S4**.

1. $(\Box A \rightarrow \Box B) \vee (\Box B \rightarrow \Box A)$, an instance of tautology $(p \rightarrow q) \vee (q \rightarrow p)$;

2. $\Box A \rightarrow A$, reflexivity axiom;
3. $\Box B \rightarrow B$, reflexivity axiom;
4. $(\Box A \rightarrow \Box B) \rightarrow (\Box A \rightarrow B)$, from 3, by propositional logic;
5. $(\Box B \rightarrow \Box A) \rightarrow (\Box B \rightarrow A)$, from 2, by propositional logic;
6. $(\Box A \rightarrow B) \vee (\Box B \rightarrow A)$, from 1,4,5, by propositional logic.

h) Not provable. Countermodel: $W = \{0, 1\}$, R is total on W , $0 \models A$, $1 \models B$. Here $0 \not\models A \rightarrow B$ and $1 \not\models B \rightarrow A$. Hence, neither of the nodes models $\Box(A \rightarrow B) \vee \Box(B \rightarrow A)$.

i) Yes, a(n easy) theorem of **S5**.

1. $\Box \neg A \rightarrow \neg A$, reflexivity axiom;
2. $\neg \neg A \rightarrow \neg \Box \neg A$, by contrapositive;
3. $A \rightarrow \neg \Box \neg A$, stripping $\neg \neg$;
4. $\neg \Box \neg A \rightarrow \Box \neg \Box \neg A$, negative introspection;
5. $A \rightarrow \Box \neg \Box \neg A$, from 3,4, by syllogism;
6. $A \rightarrow \Box \Diamond A$, renaming $\neg \Box \neg$ as \Diamond .

Exercise 16.10. Find the realizations of the following **S4** theorems in **LP**

- a) $\Box A \wedge \Box B \rightarrow \Box(\Box A \wedge \Box B)$
- b) $\Box A \vee \Box B \rightarrow \Box(\Box A \vee \Box B)$

More exactly, find proof polynomials $t(x, y)$ and $s(x, y)$ such that **LP** proves

$$\begin{aligned} x:A \wedge y:B &\rightarrow t(x, y):(x:A \wedge y:B), \\ x:A \vee y:B &\rightarrow s(x, y):(x:A \vee y:B). \end{aligned}$$

Hint. Derive in **S4** first, then try to mimic this derivation in **LP**. Use "+" to reconcile different proof terms by the same formulas, if needed. One more example: find an explicit version of $\Box A \rightarrow \Box(B \rightarrow \Box A)$.

First, we derive this formula in **S4**:

1. $\Box A \rightarrow (B \rightarrow \Box A)$, a propositional axiom;
2. $\Box(\Box A \rightarrow (B \rightarrow \Box A))$, by Necessitation, from 1;
3. $\Box(\Box A \rightarrow (B \rightarrow \Box A)) \rightarrow (\Box \Box A \rightarrow \Box(B \rightarrow \Box A))$, distributivity modal axiom;
4. $\Box \Box A \rightarrow \Box(B \rightarrow \Box A)$, by Modus Ponens, from 2,3;
5. $\Box A \rightarrow \Box \Box A$, positive introspection axiom;
6. $\Box A \rightarrow \Box(B \rightarrow \Box A)$, by propositional logic, from 4,5.

Then we emulate this derivation in **LP** using common sense and some ingenuity.

1. $x:A \rightarrow (B \rightarrow x:A)$, a propositional axiom;
2. $c:(x:A \rightarrow (B \rightarrow x:A))$, by R1;
3. $c:(x:A \rightarrow (B \rightarrow x:A)) \rightarrow (!x:x:A \rightarrow (c!x):(B \rightarrow x:A))$, application axiom A1;
4. $!x:x:A \rightarrow (c!x):(B \rightarrow x:A)$, by Modus Ponens, from 2,3;
5. $x:A \rightarrow !x:x:A$, proof checking axiom A2;
6. $x:A \rightarrow (c!x):(B \rightarrow x:A)$, by propositional logic.

Solution.

- a) 1. $\Box A \rightarrow (\Box B \rightarrow (\Box A \wedge \Box B))$, a propositional axiom
2. $\Box(\Box A \rightarrow (\Box B \rightarrow (\Box A \wedge \Box B)))$, by Necessitation
3. $\Box \Box A \rightarrow (\Box \Box B \rightarrow \Box(\Box A \wedge \Box B))$, by Distribution, twice
4. $\Box A \rightarrow \Box \Box A$, $\Box B \rightarrow \Box \Box B$, transitivity axioms
5. $\Box A \rightarrow (\Box B \rightarrow \Box(\Box A \wedge \Box B))$, from 3,4, by propositional logic
6. $\Box A \wedge \Box B \rightarrow \Box(\Box A \wedge \Box B)$, from 5, by propositional logic

The corresponding **LP**-derivation is

- a) 1. $x:A \rightarrow (y:B \rightarrow (x:A \wedge y:B))$, a propositional axiom

2. $c:(x:A \rightarrow (y:B \rightarrow (x:A \wedge y:B)))$, by R1
3. $!x:x:A \rightarrow (!y:y:B \rightarrow (c \cdot !x \cdot !y):(x:A \wedge y:B))$, by Distribution, twice
4. $x:A \rightarrow !x:x:A$, $y:B \rightarrow !y:y:B$, proof checker axioms
5. $x:A \rightarrow (y:B \rightarrow (c \cdot !x \cdot !y):(x:A \wedge y:B))$, from 3,4, by propositional logic
6. $x:A \wedge y:B \rightarrow (c \cdot !x \cdot !y):(x:A \wedge y:B)$, from 5, by propositional logic

b) cf. Lecture 11.

Exercise 16.11. Given a derivation of $A \wedge !u : u : B$ in **LP** from hypotheses $A, u : B$, find a proof polynomial $g(x, y)$ and a derivation of $g(x, y) : (A \wedge !u : u : B)$ in **LP** from hypotheses $x : A, y : u : B$.

1. A , a hypothesis
2. $u : B$, a hypothesis
3. $u : B \rightarrow !u : u : B$, a proof checking axiom of **LP**
4. $!u : u : B$, by MP from 2,3
5. $A \rightarrow (!u : u : B \rightarrow (A \wedge !u : u : B))$, a propositional axiom
6. $!u : u : B \rightarrow (A \wedge !u : u : B)$, by MP, from 1,5
7. $A \wedge !u : u : B$, by MP, from 4,6

Solution. The general rule in such sort of problems is “prefix each hypothesis by a fresh variable, each axiom with a fresh constant, and use “ \cdot ” as internal Modus Ponens”. Here is the desired derivation in **LP**:

1. $x : A$, a hypothesis
2. $y : u : B$, a hypothesis
3. $a : (u : B \rightarrow !u : u : B)$, a Necessitation of proof checking axiom of **LP** by constant a
4. $(a \cdot y) !u : u : B$, by axiom A1 of **LP**, from 2,3
5. $b : (A \rightarrow (!u : u : B \rightarrow (A \wedge !u : u : B)))$, Necessitation of a propositional axiom of **LP** by constant b
6. $(b \cdot x) : (u : u : B \rightarrow (A \wedge !u : u : B))$, by axiom A1 of **LP**, from 1,5
7. $((b \cdot x) \cdot (a \cdot y)) : (A \wedge !u : u : B)$, by axiom A1 of **LP**, from 4,6

Exercise 16.12. Given a proof in **S4** find its realization in **LP**. Feel free to use the Necessitation Rule for **LP**:

if $\mathbf{LP} \vdash F$ then $\mathbf{LP} \vdash g : F$ for some ground proof polynomial g

Solution.

- a) $\Box A \rightarrow \Box(B \rightarrow A)$
1. $A \rightarrow (B \rightarrow A)$, an axiom;
 2. $\Box(A \rightarrow (B \rightarrow A))$, by necessitation;
 3. $\Box A \rightarrow \Box(B \rightarrow A)$, by distribution

A derivation in **LP**:

1. $A \rightarrow (B \rightarrow A)$, a propositional axiom;
2. $c : (A \rightarrow (B \rightarrow A))$, Necessitation of 1 by constant c
3. $x : A \rightarrow (c \cdot x) : (B \rightarrow A)$, by axiom A1 of **LP**

- b) $\Box A \rightarrow \Box(B \rightarrow \Box A)$
1. $\Box A \rightarrow (B \rightarrow \Box A)$, an axiom;
 2. $\Box(\Box A \rightarrow (B \rightarrow \Box A))$, by necessitation;
 3. $\Box\Box A \rightarrow \Box(B \rightarrow \Box A)$, by distribution;
 4. $\Box A \rightarrow \Box\Box A$, transitivity axiom;
 5. $\Box A \rightarrow \Box(B \rightarrow \Box A)$, by syllogism, from 3,4

A derivation in **LP**:

1. $x : A \rightarrow (B \rightarrow x : A)$, a propositional axiom;

2. $c:(x:A \rightarrow (B \rightarrow A))$, Necessitation of 1 by constant c
3. $z:x:A \rightarrow (c \cdot z):(B \rightarrow x:A)$, by axiom A1 of **LP**, from 2
4. $x:A \rightarrow !x:x:A$, proof checking axiom;
- 3'. $!x:x:A \rightarrow (c \cdot !x):(B \rightarrow x:A)$, we specify z as $!x$ in 3
5. $x:A \rightarrow (c \cdot !x):(B \rightarrow x:A)$, by syllogism, from 3',4

c) $(\Box A \vee \Box \neg B) \rightarrow \Box(B \rightarrow \Box A)$

1. $\Box A \rightarrow \Box(B \rightarrow \Box A)$, as above;
2. $\neg B \rightarrow (B \rightarrow \Box A)$, in propositional logic;
3. $\Box(\neg B \rightarrow (B \rightarrow \Box A))$, by necessitation;
4. $\Box \neg B \rightarrow \Box(B \rightarrow \Box A)$, by distribution;
5. $(\Box A \vee \Box \neg B) \rightarrow \Box(B \rightarrow \Box A)$, from 1,4 in

Here is a corresponding **LP**-derivation:

1. $x:A \rightarrow (c \cdot !x):(B \rightarrow x:A)$, as above;
2. $\neg B \rightarrow (B \rightarrow x:A)$, in propositional logic;
3. $g:(\neg B \rightarrow (B \rightarrow x:A))$, by Necessitation, for some ground term g
4. $y:\neg B \rightarrow (g \cdot y):(B \rightarrow x:A)$, by axiom A1 of **LP**, from 3
5. $(x:A \vee y:\neg B) \rightarrow (c \cdot !x + g \cdot y):(B \rightarrow x:A)$, from 1,4 in