Computational Logic, Fall 2002	**Handout 16**
Practice kit for Test 3	**December 4, 2002**

Make sure you have mastered all the ideas contained in exercises from HW7 and HW8.

Exercise 16.1.

a) Derive in $_{\rightarrow}$**IntH**: $((B \rightarrow A) \rightarrow A) \rightarrow A \vdash B \rightarrow A$. Feel free to use the Deduction Theorem.

b) Internalize the deduction from (a) as a typed combinatory term. Feel free to use emulated λ-abstraction. There is no need to unfold it back to the standard combinatory format, unless you have to.

c) Check whether the combinatory term from (b) is normal.

Exercise 16.2. Find a combinator **2** such that $\mathbf{2}FA = F(FA)$ for all combinators F and A. If you use λ-abstraction, convert the result into a purely combinatory form. Is **2** a normal form?

Exercise 16.3. Write completely a λ-term whose normal form is the Church numeral $\overline{100}$.

Exercise 16.4. Prove that all the inclusions are strict: $\mathbf{K} \subset \mathbf{K4} \subset \mathbf{S4} \subset \mathbf{S5}$. In practical terms it suffices to show that

$\mathbf{K} \not\vdash \Box p \rightarrow \Box\Box p$,

$\mathbf{K4} \not\vdash \Box p \rightarrow p$,

$\mathbf{S4} \not\vdash \neg\Box p \rightarrow \Box\neg\Box p$.

Exercise 16.5. Which of the following is provable in **S4**? in **S5**?

a) $\Box(p \vee \neg p)$

b) $\Box p \vee \Box\neg p$

c) $\Box p \vee \neg\Box p$

d) $\Box p \vee \Box\neg\Box p$

e) $(\Box A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$

f) $\Box(A \rightarrow B) \rightarrow \Box(\Box A \rightarrow \Box B)$

g) $\Box(A \rightarrow B) \rightarrow (A \rightarrow \Box B)$

h) $(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

Exercise 16.6. Show that **S5**=**S4**+$\Box(A \rightarrow \neg\Box B) \rightarrow (A \rightarrow \Box\neg\Box B)$. Basically, you have to demonstrate that both

a) $\mathbf{S5} \vdash \Box(A \rightarrow \neg\Box B) \rightarrow (A \rightarrow \Box\neg\Box B)$ and

b) $\mathbf{S4}+\Box(A \rightarrow \neg\Box B) \rightarrow (A \rightarrow \Box\neg\Box B) \vdash \neg\Box A \rightarrow \Box\neg\Box A$.

Exercise 16.7. Prove in **S4**:

a) $\Box A \rightarrow \Box(B \rightarrow A)$

b) $\Box A \rightarrow \Box(B \rightarrow \Box A)$

c) $(\Box A \vee \Box\neg B) \rightarrow \Box(B \rightarrow \Box A)$

d) $\Box\neg B \rightarrow \Box(\Box B \rightarrow A)$

Exercise 16.8. Show that the following formulas are not derivable in **S4**. To achieve that find an **S4**-countermodel (i.e. reflexive and transitive countermodel Kripke) for this formula.

a) $\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$

b) $\Box(\Box A \rightarrow A) \rightarrow \Box A$.

c) $(\Box A \vee \neg\Box B) \rightarrow \Box(\Box B \rightarrow \Box A)$

d) $A \rightarrow \Box\Diamond A$

Exercise 16.9. Which of the following is provable in **S5**? Give a proof, if any. Provide an-**S5** countermodel (i.e. symmetric, reflexive and transitive countermodel Kripke) for a formula.

a) $\Box(A \to B) \leftrightarrow (\Box A \to \Box B)$
b) $\Box A \leftrightarrow \Box\Box A$
c) $A \to \Box A$
d) $(\Box A \to \Box B) \vee (\Box B \to \Box A)$
e) $\Box(\Box A \to B) \vee \Box(\Box B \to A)$
f) $\Box(\Box A \to A) \to \Box A$
g) $(\Box A \to B) \vee (\Box B \to A)$
h) $\Box(A \to B) \vee \Box(B \to A)$
i) $A \to \Box\Diamond A$

Exercise 16.10. Find the realizations of the following **S4** theorems in **LP**

a) $\Box A \wedge \Box B \to \Box(\Box A \wedge \Box B)$
b) $\Box A \vee \Box B \to \Box(\Box A \vee \Box B)$

More exactly, find proof polynomials $t(x, y)$ and $s(x, y)$ such that **LP** proves
$x{:}A \wedge y{:}B \to t(x, y){:}(x{:}A \wedge y{:}B)$,
$x{:}A \vee y{:}B \to s(x, y){:}(x{:}A \vee y{:}B)$.

Hint. Derive in **S4** first, then try to mimic this derivation in **LP**. Use "+" to reconcile different proof terms by the same formulas, if needed. One more example: find an explicit version of $\Box A \to \Box(B \to \Box A)$. First, we derive this formula in **S4**:
1. $\Box A \to (B \to \Box A)$, a propositional axiom;
2. $\Box(\Box A \to (B \to \Box A))$, by Necessitation, from 1;
3. $\Box(\Box A \to (B \to \Box A)) \to (\Box\Box A \to \Box(B \to \Box A))$, distributivity modal axiom;
4. $\Box\Box A \to \Box(B \to \Box A)$, by Modus Ponens, from 2,3;
5. $\Box A \to \Box\Box A$, positive introspection axiom;
6. $\Box A \to \Box(B \to \Box A)$, by propositional logic, from 4,5.

Then we emulate this derivation in **LP** using common sense and some ingenuity.
1. $x{:}A \to (B \to x{:}A)$, a propositional axiom;
2. $c{:}(x{:}A \to (B \to x{:}A))$, by R1;
3. $c{:}(x{:}A \to (B \to x{:}A)) \to (!x{:}x{:}A \to (c\cdot !x){:}(B \to x{:}A))$, application axiom A1;
4. $!x{:}x{:}A \to (c\cdot !x){:}(B \to x{:}A)$, by Modus Ponens, from 2,3;
5. $x{:}A \to !x{:}x{:}A$, proof checking axiom A2;
6. $x{:}A \to (c\cdot !x){:}(B \to x{:}A)$, by propositional logic,

Exercise 16.11. Given a derivation of $A \wedge !u : u : B$ in **LP** from hypotheses $A, u : B$, find a proof polynomial $g(x, y)$ and a derivation of $g(x, y){:}(A \wedge !u{:}u{:}B)$ in **LP** from hypotheses $x{:}A, y{:}u{:}B$.
1. A, a hypothesis
2. $u{:}B$, a hypothesis
3. $u{:}B \to !u{:}u{:}B$, a proof checking axiom of **LP**
4. $!u{:}u{:}B$, by MP from 2,3
5. $A \to (!u{:}u{:}B \to (A \wedge !u{:}u{:}B))$, a propositional axiom
6. $!u{:}u{:}B \to (A \wedge !u{:}u{:}B)$, by MP, from 1,5
7. $A \wedge !u{:}u{:}B$, by MP, from 4,6

Exercise 16.12. Given a proof in **S4** find its realization in **LP**. Feel free to use the Necessitation Rule for **LP**:

if **LP** $\vdash F$ *then* **LP** $\vdash g{:}F$ *for some ground proof polynomial g*

a) $\Box A \to \Box(B \to A)$
1. $A \to (B \to A)$, an axiom;

2. $\Box(A \to (B \to A))$, by necessitation;
3. $\Box A \to \Box(B \to A)$, by distribution

b) $\Box A \to \Box(B \to \Box A)$

1. $\Box A \to (B \to \Box A)$, an axiom;
2. $\Box(\Box A \to (B \to \Box A))$, by necessitation;
3. $\Box\Box A \to \Box(B \to \Box A)$, by distribution;
4. $\Box A \to \Box\Box A$, transitivity axiom;
5. $\Box A \to \Box(B \to \Box A)$, by syllogism, from 3,4

c) $(\Box A \vee \Box\neg B) \to \Box(B \to \Box A)$

1. $\Box A \to \Box(B \to \Box A)$, as above;
2. $\neg B \to (B \to \Box A)$, in propositional logic;
3. $\Box(\neg B \to (B \to \Box A))$, by necessitation;
4. $\Box\neg B \to \Box(B \to \Box A)$, by distribution;
5. $(\Box A \vee \Box\neg B) \to \Box(B \to \Box A)$, from 1,4 in propositional logic.