Computational Logic, Fall 2002	**Handout 11**
Practice kit for Test 2	**November 6, 2002**

Make sure you have mastered all the ideas contained in exercises from HW4, HW5 and HW6.

Exercise 11.1. Consider the $\{\to, \wedge\}$ fragment of intuitionistic logic ${}_{\to,\wedge}\mathbf{IntH}$ in the language $\{\to, \wedge\}$ obtained from ${}_{\to}\mathbf{IntH}$ by adding axioms concerning $\wedge$:

A3. $A \wedge B \to A$,
A4. $A \wedge B \to B$,
A5. $A \to (B \to (A \wedge B)$.

The same fragment in the sequent format ${}_{\to,\wedge}\mathbf{IntG}$ is obtained from ${}_{\to}\mathbf{IntG}$ by adding rules about $\wedge$:

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\,(\wedge, \Rightarrow) \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\,(\wedge, \Rightarrow) \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\,(\Rightarrow, \wedge)$$

Systems ${}_{\to,\wedge}\mathbf{IntH}$ and ${}_{\to,\wedge}\mathbf{IntG}$ are equivalent: $A_1, A_2, \ldots, A_n \Rightarrow B$ is provable in ${}_{\to,\wedge}\mathbf{IntG}$ if and only if $A_1 \wedge A_2 \wedge \ldots \wedge A_n \vdash B$ in ${}_{\to,\wedge}\mathbf{IntH}$.

a) Establish the above equivalence. Since we have done this for the implicational fragment of **Int** already, it suffices to check axioms and rules about $\wedge$ only. In particular, find ${}_{\to,\wedge}\mathbf{IntG}$-proofs of A3, A4, A5, and ${}_{\to,\wedge}\mathbf{IntH}$ proofs of rules $(\wedge, \Rightarrow)$, $(\Rightarrow, \wedge)$.

Solution. An easy inspection of the corresponding proof of equivalence of **IntH** and **IntG** shows that emulating $\wedge$-axioms of **IntH** needs only $\wedge$-rules of **IntG** and vice versa.

b) Establish cut-elimination theorem for ${}_{\to,\wedge}\mathbf{IntG}$. Actually, this fact is an easy corollary of cut-elimination for the whole **IntG** and a subformula property of cut-free proofs.

Solution. Suppose a sequent α in the language $\to, \wedge$ is proved in ${}_{\to,\wedge}\mathbf{IntG}$. Then there is a cut-free proof $\mathcal{D}$ of α in **IntG**. By the subformula property, such a proof contains only rules for connectives explicitly occurring in the final sequent α. In particular, in $\mathcal{D}$ there are only rules concerning $\to$ or $\wedge$. Hence, $\mathcal{D}$ is in fact a (cut-free) derivation of α in ${}_{\to,\wedge}\mathbf{IntG}$.

c) Establish conservativity of **IntH** with respect to ${}_{\to,\wedge}\mathbf{IntH}$. It suffices to show that for any $\{\to, \wedge\}$-formula F if **IntH** proves F then ${}_{\to,\wedge}\mathbf{IntH}$ proves F.

Solution. It suffices to establish that for a $\to, \wedge$-formula F if **IntH** proves F then F is already provable in ${}_{\to,\wedge}\mathbf{IntH}$. Suppose **IntH** proves such a formula F. Then **IntG** proves $\Rightarrow F$. By cut elimination property, there is a cut-free derivation $\mathcal{D}$ of $\Rightarrow F$ in **IntG**. By the same argument as in (b), $\mathcal{D}$ is already a derivation in ${}_{\to,\wedge}\mathbf{IntG}$. By (a), F is provable in ${}_{\to,\wedge}\mathbf{IntH}$.

Exercise 11.2. Apply the step-by-step algorithm of transforming Gentzen style derivations into **IntN**-derivations from Lecture 5 (cf. also Troelstra and Schwichtenberg, section 3.3) to transform a derivation of a sequent $\Rightarrow \neg\neg(p \vee \neg p)$ into a natural derivation of a formula $\neg\neg(p \vee \neg p)$.

Solution. Consider a Gentzen style derivation of $\neg\neg(p \vee \neg p)$:

$$\begin{array}{c} p \Rightarrow p \\ \hline p \Rightarrow p \lor \neg p \\ \hline \neg(p \lor \neg p), p \Rightarrow \\ \hline \neg(p \lor \neg p) \Rightarrow \neg p \\ \hline \neg(p \lor \neg p) \Rightarrow p \lor \neg p \\ \hline \neg(p \lor \neg p), \neg(p \lor \neg p) \Rightarrow \\ \hline \neg(p \lor \neg p) \Rightarrow \\ \hline \Rightarrow \neg\neg(p \lor \neg p) \end{array}$$

This derivation should be transformed into one with **nonempty succedents**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{p \Rightarrow p \lor \neg p} \qquad p, \bot \Rightarrow \bot}{\neg(p \lor \neg p), p \Rightarrow \bot}}{\neg(p \lor \neg p) \Rightarrow \neg p}}{\neg(p \lor \neg p) \Rightarrow p \lor \neg p} \qquad \neg(p \lor \neg p), \bot \Rightarrow \bot}{\neg(p \lor \neg p), \neg(p \lor \neg p) \Rightarrow \bot}}{\neg(p \lor \neg p) \Rightarrow \bot}}{\Rightarrow \neg\neg(p \lor \neg p)}$$

Use algorithm of transforming **IntG** proofs with nonempty succedents to **IntN** proofs (T& S 3.3.1) to obtain the desired natural derivation of $\neg\neg(p \lor \neg p)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p]^u}{p \lor \neg p} \qquad [\neg(p \lor \neg p)]^v}{\bot}}{p \to \bot}\,u}{p \lor \neg p} \qquad [\neg(p \lor \neg p)]^v}{\bot}}{\neg(p \lor \neg p) \to \bot}\,v$$

Exercise 11.3. Give a derivation of $(\neg p \lor q) \to (p \to q)$ in **IntN**. Apply the algorithm of transforming **IntN**-derivations into Gentzen style derivations from Lecture 5 (cf. also Troelstra and Schwichtenberg, section 3.3) to transform this derivation into an **IntG**-derivation of a sequent $\Rightarrow (\neg p \lor q) \to (p \to q)$.

Solution. We first have to complete the description of alrorithm translating **IntN** derivations to **IntG** derivations, which was left in T&S 3.3.1A as an exercise. Actually, we have to consider the rules concerning $\lor$.

$$\dfrac{\begin{array}{c}\mathcal{D} \\ A\end{array}}{A \lor B} \qquad \text{goes to} \qquad \dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \lor B}$$

$$\dfrac{A \vee B \qquad \begin{array}{c}[A]^u\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[B]^u\\ \vdots\\ C\end{array}}{C} \qquad \text{goes to} \qquad \dfrac{\Gamma, A \Rightarrow C \quad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}$$

Now we start with the natural derivation of $(\neg p \vee q) \to (p \to q)$:

$$\dfrac{\dfrac{\dfrac{[\neg p \vee q]^w \qquad \dfrac{\dfrac{[p]^v \quad [\neg p]^u}{\bot}}{q} \qquad [q]^u}{q}\, u}{p \to q}\, v}{(\neg p \vee q) \to (p \to q)}\, w$$

We anticipate two Cut rules in the corresponding **G**-derivation, since each of the elimination **N**-rules under the standard **G**-translation (3.3.1 in T&S) produces a Cut. In particular, the **N**- part

$$\dfrac{p \quad p \to \bot}{\bot} \qquad \text{goes to} \qquad \dfrac{p \to \bot \Rightarrow p \to \bot \qquad \dfrac{p \Rightarrow p \quad p, \bot \Rightarrow \bot}{p \to \bot, p \Rightarrow \bot}}{p, p \to \bot \Rightarrow \bot}\, \text{Cut}$$

Of course, there are a lot of redundancies in the resulting **G**-derivation, but this is how the translation algorithm works. **Do not simplify the resulting G-derivation in this solution!** The desired **G** derivation is then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \to \bot \Rightarrow p \to \bot \qquad \dfrac{p \Rightarrow p \quad p, \bot \Rightarrow \bot}{p \to \bot, p \Rightarrow \bot}}{p, p \to \bot \Rightarrow \bot}\, \text{Cut} \qquad \bot \Rightarrow q}{p, p \to \bot \Rightarrow q}\, \text{Cut} \qquad q \Rightarrow q}{\neg p \vee q, p \Rightarrow q}}{\neg p \vee q \Rightarrow p \to q}}{\Rightarrow (\neg p \vee q) \to (p \to q)}$$

Exercise 11.4. Find a λ-term derivation corresponding to an **IntN**-derivation of

$$(p \to (q \wedge r)) \to ((p \to q) \wedge (p \to r)).$$

Solution. First, we have to find an **IntN**-derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p \to (q \wedge r)]^x \quad [p]^y}{q \wedge r}}{q}}{p \to q}\, y \qquad \dfrac{\dfrac{\dfrac{[p \to (q \wedge r)]^x \quad [p]^z}{q \wedge r}}{r}}{p \to r}\, z}{(p \to q) \wedge (p \to r)}}{(p \to (q \wedge r)) \to (p \to q) \wedge (p \to r)}\, x$$

Now the corresponding λ-term:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x:p\to q\wedge r \qquad y:p}{(xy):q\wedge r}}{\mathbf{p}_0(xy):q}}{\lambda y.\mathbf{p}_0(xy):p\to q} \qquad \dfrac{\dfrac{\dfrac{x:p\to q\wedge r \qquad z:p}{(xz):q\wedge r}}{\mathbf{p}_1(xz):r}}{\lambda z.\mathbf{p}_1(xz):p\to r}}{\mathbf{p}(\lambda y.\mathbf{p}_0(xy),\lambda z.\mathbf{p}_1(xz)):(p\to q)\wedge(p\to r)}}{\lambda x.\mathbf{p}(\lambda y.\mathbf{p}_0(xy),\lambda z.\mathbf{p}_1(xz)):(p\to q\wedge r)\to(p\to q)\wedge(p\to r)}$$

Exercise 11.5. Find derivations in $_{\to}$**IntH**:

a) $A\to(A\to B)\vdash A\to B$

Solution. We will write XY for $X\to Y$, XYZ for $X\to(Y\to Z)$, etc., for short.

1. AAB hypothesis
2. $AAB\to(AA\to AB)$, axiom 2
3. $AA\to AB$, MP
4. AA, well known theorem, T&S 1.3.6.
5. AB, MP

b) $B\to C\vdash(A\to B)\to(A\to C)$

Solution.

1. BC, hypothesis
2. $BC\to ABC$, axiom 1
3. ABC, MP
4. $ABC\to(AB\to AC)$, axiom 2
5. $AB\to AC$, MP

Exercise 11.6. Find internalization of the proofs from 11.5 as combinatory terms.

Solution. a) We fist recall that a combinatory term corresponding to the standard derivation of $A\to A$ (T&S 1.3.6) is $\mathbf{s}_1\mathbf{k}_1\mathbf{k}_2$, where

$\mathbf{s}_1:(A(AA)A)\to(AAA\to AA)$
$\mathbf{k}_1:A(AA)A$
$\mathbf{k}_2:AAA$

By the format of the question, we are looking for a combinatory term $t(x)$ of type $A\to B$ depending on a variable $x^{A\to(A\to B)}$. We build $t(x)$ step-by-step following the derivation from 11.5a.

1. $x:AAB$
2. $\mathbf{s}_2:AAB\to(AA\to AB)$
3. $(\mathbf{s}_2\cdot x):AA\to AB$
4. $(\mathbf{s}_1\mathbf{k}_1\mathbf{k}_2):AA$
5. $\mathbf{s}_2x(\mathbf{s}_1\mathbf{k}_1\mathbf{k}_2):AB$

Hence $t(x)$ is $\mathbf{s}_2x(\mathbf{s}_1\mathbf{k}_1\mathbf{k}_2)$.

Solution of 11.6b.

1. $x:BC$, hypothesis
2. $\mathbf{k}:BC\to ABC$, axiom 1
3. $\mathbf{k}x:ABC$, MP
4. $\mathbf{s}:ABC\to(AB\to AC)$, axiom 2
5. $\mathbf{s}(\mathbf{k}x):AB\to AC$, MP

Exercise 11.7. Let $\mathbf{I}^{A\to A}$ be the identity combinator

$$\mathbf{s}^{(A\to((A\to A)\to A))\to((A\to(A\to A))\to(A\to A))}\mathbf{k}^{A\to((A\to A)\to A)}\mathbf{k}^{A\to(A\to A)}$$

Find normal forms of the following combinatory terms. Types are suppressed for short, assume they all match properly, x, y, z are variables.

a) $\mathbf{I}$

b) $\mathbf{I}x$.

From here note, that it is possible for terms u and v to be both normal whereas $u \cdot v$ is not normal!

c) $\mathbf{ss}(\mathbf{kI})$

d) $\mathbf{ss}(\mathbf{kI})x$

e) $\mathbf{ss}(\mathbf{kI})xy$

Solution.

a) $\mathbf{I} = \mathbf{skk}$ - a normal form

b) $\mathbf{I}x = \mathbf{skk}x \succeq \mathbf{k}x(\mathbf{k}x) \succeq x$ - a normal form.

c) $\mathbf{ss}(\mathbf{kI})$ - a normal form

d) $\mathbf{ss}(\mathbf{kI})x \succeq \mathbf{s}x(\mathbf{kI}x) \succeq \mathbf{s}x\mathbf{I} = \mathbf{s}x(\mathbf{skk})$ - a normal form

e) $\mathbf{ss}(\mathbf{kI})xy \succeq \mathbf{s}x\mathbf{I}y \succeq xy(\mathbf{I}y) \succeq xyy$ - a normal form.