Computational Logic, Fall 2002	**Handout 11**
Practice kit for Test 2	**November 6, 2002**

Make sure you have mastered all the ideas contained in exercises from HW4, HW5 and HW6.

Exercise 11.1. Consider the $\{\to,\wedge\}$ fragment of intuitionistic logic ${}_{\to,\wedge}$**IntH** in the language $\{\to,\wedge\}$ obtained from ${}_{\to}$**IntH** by adding axioms concerning $\wedge$:

A3. $A\wedge B\to A$,
A4. $A\wedge B\to B$,
A5. $A\to(B\to(A\wedge B)$.

The same fragment in the sequent format ${}_{\to,\wedge}$**IntG** is obtained from ${}_{\to}$**IntG** by adding rules about $\wedge$:

$$\frac{A,\Gamma\Rightarrow\Delta}{A\wedge B,\Gamma\Rightarrow\Delta}\,(\wedge,\Rightarrow)\qquad\frac{B,\Gamma\Rightarrow\Delta}{A\wedge B,\Gamma\Rightarrow\Delta}\,(\wedge,\Rightarrow)\qquad\frac{\Gamma\Rightarrow A\quad\Gamma\Rightarrow B}{\Gamma\Rightarrow A\wedge B}\,(\Rightarrow,\wedge)$$

Systems ${}_{\to,\wedge}$**IntH** and ${}_{\to,\wedge}$**IntG** are equivalent: $A_1,A_2,\ldots,A_n\Rightarrow B$ is provable in ${}_{\to,\wedge}$**IntG** if and only if $A_1\wedge A_2\wedge\ldots\wedge A_n\vdash B$ in ${}_{\to,\wedge}$**IntH**.

a) Establish the above equivalence. Since we have done this for the implicational fragment of **Int** already, it suffices to check axioms and rules about $\wedge$ only. In particular, find ${}_{\to,\wedge}$**IntG**-proofs of A3, A4, A5, and ${}_{\to,\wedge}$**IntH** proofs of rules $(\wedge,\Rightarrow)$, $(\Rightarrow,\wedge)$.

b) Establish cut-elimination theorem for ${}_{\to,\wedge}$**IntG**. Actually, this fact is an easy corollary of cut-elimination for the whole **IntG** and a subformula property of cut-free proofs.

c) Establish conservativity of **IntH** with respect to ${}_{\to,\wedge}$**IntH**. It suffices to show that for any $\{\to,\wedge\}$-formula F if **IntH** proves F then ${}_{\to,\wedge}$**IntH** proves F.

Exercise 11.2. Apply the step-by-step algorithm of transforming Gentzen style derivations into **IntN**-derivations from Lecture 5 (cf. also Troelstra and Schwichtenberg, section 3.3) to transform a derivation of a sequent $\Rightarrow\neg\neg(p\vee\neg p)$ into a natural derivation of a formula $\neg\neg(p\vee\neg p)$.

Exercise 11.3. Give a derivation of $(\neg p\vee q)\to(p\to q)$ in **IntN**. Apply the algorithm of transforming **IntN**-derivations into Gentzen style derivations from Lecture 5 (cf. also Troelstra and Schwichtenberg, section 3.3) to transform this derivation into an **IntG**-derivation of a sequent $\Rightarrow(\neg p\vee q)\to(p\to q)$.

Exercise 11.4. Find a λ-term derivation corresponding to an **IntN**-derivation of

$$(p\to(q\wedge r))\to((p\to q)\wedge(p\to r)).$$

Exercise 11.5. Find derivations in ${}_{\to}$**IntH**:

a) $A\to(A\to B)\vdash A\to B$
b) $B\to C\vdash(A\to B)\to(A\to C)$

Exercise 11.6. Find internalization of the proofs from 11.5 as combinatory terms.

Exercise 11.7. Let $\mathbf{I}^{A\to A}$ be the identity combinator

$$\mathbf{s}^{(A\to((A\to A)\to A))\to((A\to(A\to A))\to(A\to A))}\mathbf{k}^{A\to((A\to A)\to A)}\mathbf{k}^{A\to(A\to A)}$$

Find normal forms of the following combinatory terms. Types are suppressed for short, assume they all match properly, x, y, z are variables.

a) $\mathbf{I}$

b) $\mathbf{I}x$.

From here note, that it is possible for terms u and v to be both normal whereas $u \cdot v$ is not normal!

c) $\mathbf{ss}(\mathbf{kI})$

d) $\mathbf{ss}(\mathbf{kI})x$

e) $\mathbf{ss}(\mathbf{kI})xy$