CL 2002:

Computational Logic

(*Lecture 4*)

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This lecture plan

1. Some applications of the Cut Elimination Theorem.
2. Equivalent forms of Cut Rule, two parameters complexity measure.
3. System of reductions, convergence.
4. Kripke completeness theorem for Int.

Need of Cut? As we have noticed, a Cut Rule has been used in emulating Modus Ponens in **IntG**. On the other hand, observations demonstrate that Cuts can be eliminated from particular proofs in **IntG**.

$$\dfrac{\dfrac{A \Rightarrow A}{A \wedge B \Rightarrow A} \qquad \dfrac{A \Rightarrow A}{A \Rightarrow A \vee B}}{A \wedge B \Rightarrow A \vee B}\ \text{Cut}$$

$$\dfrac{\dfrac{A \Rightarrow A}{A \wedge B \Rightarrow A}}{A \wedge B \Rightarrow A \vee B} \qquad \text{No Cuts}$$

Another example where Cut is present:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{A \Rightarrow A}{A \Rightarrow A \lor B}
\qquad
\dfrac{C \Rightarrow C}{A, C \Rightarrow C}
}{A \lor B \to C, A \Rightarrow C}
}{A \lor B \to C \Rightarrow A \to C}
\qquad
\dfrac{
\dfrac{
\dfrac{
A \Rightarrow A
\qquad
\dfrac{C \Rightarrow C}{A, C \Rightarrow C}
}{A \to C, A \Rightarrow C}
}{A \to C, A \land B \Rightarrow C}
}{A \to C \Rightarrow A \land B \to C}
}{A \lor B \to C \Rightarrow A \land B \to C}\ \text{Cut}
$$

but can be eliminated:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A \vee B} \qquad \dfrac{C \Rightarrow C}{A, C \Rightarrow C}}{A \vee B \rightarrow C, A \Rightarrow C}}{A \vee B \rightarrow C, A \wedge B \Rightarrow C}}{A \vee B \rightarrow C \Rightarrow A \wedge B \rightarrow C}$$

No Cuts

Why would one need to eliminate Cuts? Here is a nice corollary of the Cut Elimination Theorem, much more are yet to come.

Disjunctive Property for intuitionistic logic

If **IntG** *proves* $\Rightarrow A \vee B$*, then* **IntG** *proves* $\Rightarrow A$ *or* **IntG** *proves* $\Rightarrow B$

Proof. Suppose **IntG** proves $\Rightarrow A \vee B$ without Cuts. Consider the very last rule in this derivation. A direct inspection of rules of **IntG** leaves only three possibilities

$$\frac{\Rightarrow}{\Rightarrow A \vee B}\ \text{Weakening}, \qquad \frac{\Rightarrow A}{\Rightarrow A \vee B}\ (\Rightarrow, \vee), \qquad \frac{\Rightarrow B}{\Rightarrow A \vee B}\ (\Rightarrow, \vee),$$

first of which is impossible since the empty sequent $\Rightarrow$ is not derivable without Cut (answer, why).

Cut Elimination Theorem

Any proof in **IntG** *possibly containing Cuts can be reduces by a proper sequence of proof transformations to a Cut-free proof of the same sequent.*

Proof. "Basic Proof Theory" book, Chapter 4.

Kripke Completeness Theorem for intuitionistic logic.

$$\Gamma \vdash F \quad \textit{iff} \quad \Gamma \models F$$

Proof. Soundness:

$$\textit{If } \Gamma \vdash F \ \ \textit{then} \ \ \Gamma \models F$$

has already been established in Lecture 2. Completeness:

$$\textit{If } \Gamma \models F \quad \textit{then} \quad \Gamma \vdash F$$

is an easy corollary of the following

Finite Model Property of **Int**:

If $\Gamma \nvdash F$*, then there is a finite Kripke countermodel for* F*.*