CL 2002:

Computational Logic

(*Lecture 3*)

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This lecture plan

1. Sequents

2. Gentzen style proof system **IntG** for intuitionistic logic.

3. Examples of derivations in **IntG**

4. Equivalence of **Int** and **IntG**.

5. Cut free derivations in **IntG** as a basis for an automated proof search.

Sequents. In a general setting a sequent is a figure $\Gamma \Rightarrow \Delta$, where Γ, Δ are finite multisets of formulas, i.e. the number of occurrences of a formula in a multiset counts, but not the order.

Example: $\{p, p, \bot, \bot, \bot, q, q, q\}$ and $\{p, \bot, q\}$ are equal as sets but different as multisets.

The idea behind an intuitionistic sequent (i.e. Gentzen style) calculus is that a sequent $\Gamma \Rightarrow F$ represents a derivation from hypothesis $\Gamma \vdash F$ in the traditional Hilbert style system. The case of empty Δ is covered by a convention: a sequent $\Gamma \Rightarrow$ represents a derivation from hypothesis $\Gamma \vdash \bot$.

Intuitionistis sequent is a sequent $\Gamma \Rightarrow \Delta$, where Δ contains not more than one formula (i.e. $|\Delta| \leq 1$).

Gentzen proof systems have few simple axioms and plenty of rules. They are not good for specification purposes, but are very proof friendly and constitute basis for practically all automated deduction systems. Right now we will introduce a Gentzen style system **IntG** for propositional intuitionistic logic. This system essentially coincides with **G1i** from "Basic Proof Theory".

Axioms: all sequents of form $A \Rightarrow A$ and $\bot \Rightarrow$. It even suffices to consider *atomic* axioms only, i.e. when A is a propositional variable.

Structural rules: weakening

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ , \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}$$

contraction

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}$$

Logical rules:

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\,(\wedge, \Rightarrow) \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\,(\wedge, \Rightarrow) \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\,(\Rightarrow, \wedge)$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\,(\Rightarrow, \vee) \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\,(\Rightarrow, \vee) \qquad \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\,(\vee, \Rightarrow)$$

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}\,(\Rightarrow, \rightarrow) \qquad \frac{\Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}\,(\rightarrow, \Rightarrow)$$

Cut rule

$$\frac{\Gamma \Rightarrow A \qquad A, \Gamma' \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta}\,\text{Cut}$$

Some additional rules which can be derived in **IntG**

Negation

$$\frac{A, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg A} \qquad \frac{\Gamma \Rightarrow A}{\neg A, \Gamma \Rightarrow}$$

Proof:

$$\dfrac{\dfrac{A, \Gamma \Rightarrow}{A, \Gamma \Rightarrow \bot}\ \text{weakening}}{\Gamma \Rightarrow A \rightarrow \bot}\ (\Rightarrow, \rightarrow) \qquad \dfrac{\Gamma \Rightarrow A \qquad \dfrac{\bot \Rightarrow}{\bot, \Gamma \Rightarrow}\ \text{weakenings}}{A \rightarrow \bot, \Gamma \Rightarrow}$$

Additive Cut

$$\frac{\Gamma \Rightarrow A \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Proof: exercise!

More examples: de Morgan principle in **IntG**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \wedge B \Rightarrow A}}{\neg A, A \wedge B \Rightarrow}}{\neg A \Rightarrow \neg(A \wedge B)} \qquad \dfrac{\dfrac{\dfrac{B \Rightarrow B}{A \wedge B \Rightarrow B}}{\neg B, A \wedge B}}{\neg B \Rightarrow \neg(A \wedge B)}}{(\neg A \vee \neg B) \Rightarrow \neg(A \wedge B)}}{\Rightarrow (\neg A \vee \neg B) \rightarrow \neg(A \wedge B)}$$

Other cases of de Morgan Laws: $(\neg A \wedge \neg B) \leftrightarrow \neg(A \vee B)$ – exercise.

Playing with negation: prove $A \rightarrow \neg\neg A$, $\neg\neg\neg A \rightarrow \neg A$

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, \neg A \Rightarrow}}{A \Rightarrow \neg\neg A}}{\Rightarrow A \rightarrow \neg\neg A}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{as before}}{A \Rightarrow \neg\neg A}}{\neg\neg\neg A, A \Rightarrow}}{\neg\neg\neg A \Rightarrow \neg A}}{\Rightarrow \neg\neg\neg A \Rightarrow \neg A}$$

A failed attempt to prove $\neg\neg p \to p$

$$\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\Rightarrow p, \neg p}}{\neg\neg p \Rightarrow p}}{\Rightarrow \neg\neg p \to p}$$

The sequent in red is not an intuitionistic one. The proof above is OK from the classical standpoint (classical sequents admit multiple formulas in Δ), but is not acceptable in **IntG**.

Exercises: Try and fail to prove $p \vee \neg p$ in **IntG**. Try and succeed in proving $\neg\neg(p \vee \neg p)$ in **IntG**.

Equivalence of **Int** and **IntG**

Theorem.
A sequent $\Gamma \Rightarrow F$ *is derivable in* **IntG** *if and only if* $\Gamma \vdash F$ *in* **Int**.

Proof. Induction on derivations in **IntG** and in **Int** respectively. Details were given in class, cf. also "BPT" (Basic Proof Theory book). Note that the rule Cut is needed to imitate the Modus Ponens Step in **IntG**.

Cut rule and automated proof search.

The rule Cut is the only one in **IntG** which violates the *subformula property*: each formula occurring in the premises of a given rule is a subformula of some formula in the concluding sequent. The subformula property is essential for backward search methods in automated proving. An stunning fact first discovered by Gentzen in the early 1930s is that the Cut rule can be eliminated from any classical and intuitionistic derivation. We will establish this important fact later.